# On the Vibrations of a Planar System of Bars Cylindrically Jointed

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*Abstract:* - This paper discusses the vibrations of planar systems of bars. The main hypothesis is that the elongations are small enough in order to use linear theory. The deformations are obtained if and only if the directions of bars form a system of at least two independent directions. In the opposite case we present the non-linear theory for the determination of elongations of bars and the deviations of the connecting point. In the first situation, when the directions of bars contain at least two independent directions, one may obtain the small deviations when the system is acted by given forces. For harmonic actuations, we determined some Lissajous type figures as we presented in the numerical application that highlights the theory.

*Keywords:* - compatibility, small deviations, vibrations, equilibrium

## 1. INTRODUCTION

Systems of bars are very often met in theory and practice. The problems concerning these systems may be divided in two categories: problems in which the bars have small elongations and deviations [1 – 6], and problems in which the elongations or deviations are large enough so that the linear approximations are not valid [7 – 18]. To be out to obtain the solutions of the second type problems, authors make some assumptions concerning the lengths of the deformed bars, relation between elongations and deviations etc. [7 – 17]. Reference [18] detailed the general case with no a priori assumptions.

Fundamentals of vibrations of systems of bars are described in may references for small deviations [20 – 22]. The authors usually treat the linear case for very simple systems. Some simple non-linear situations are discussed in [23 – 27].

In Reference [28] authors discuss the vibrations of a system of two cantilever beams, connected one to another and having large elongations and deviations.

In this paper we discuss the vibrations of a planar systems of bars. First of all we determine the cases in which the theory of small deviations may be applied. For the non-compatible systems of bars we determine the non-linear systems from which one may determine the deviations. Considering the linear case we present the determination of the vibrations.

## 2. PROBLEM FORMULATION

We consider a system of $n$ planar bars, cylindrically jointed one to another at the point $O$ (Fig. 1). In Figure we represented only one bar.

Each bar is also jointed at the point $A_i$, $i=\overline{1,n}$.

For each bar we denoted by $E_i$ the elastic modulus, $A_i$ the area of the cross-section of the bar, and $l_i$ the length of the bar. Let $k_i$ be

$$k_i = \frac{E_i A_i}{l_i},\ i=\overline{1,n}. \tag{1}$$

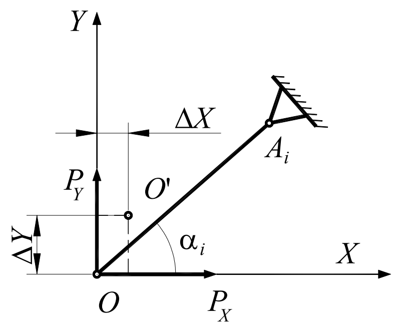


**Figure 1.** The system.

Point $O$ is acted by a force, which has the projections $P_X$ and $P_Y$ onto the axes of coordinates.

The angle between the bar $OA_i$ and the positive direction of the $OX$ - axis is denoted by $\alpha_i$.

According to [19] the deviation $\Delta X$ and $\Delta Y$ of the point $O$ are obtained from the linear system

$$\begin{cases} \left(\sum_{i=1}^{n} k_i \cos^2\alpha_i\right)\Delta X + \\ +\left(\sum_{i=1}^{n} k_i \cos\alpha_i \sin\alpha_i\right)\Delta Y = P_X , \\ \left(\sum_{i=1}^{n} k_i \cos\alpha_i \sin\alpha_i\right)\Delta X + \\ +\left(\sum_{i=1}^{n} k_i \sin^2\alpha_i\right)\Delta Y = P_Y , \end{cases} \tag{2}$$

where $n$ is the member of bars.

The formula (2) are valid only for small elongations of the bars which imply small deviations $\Delta X$ and $\Delta Y$.

## 3. COMPATIBILITY OF THE SYSTEM

### 3.1. Case of two bars

The system becomes

$$\begin{cases} \left(k_1 \cos^2\alpha_1 + k_2 \cos^2\alpha_2\right)\Delta X + \\ +\left(k_1 \cos\alpha_1 \sin\alpha_1 + k_2 \cos\alpha_2 \sin\alpha_2\right)\Delta Y = P_X, \\ \left(k_1 \cos\alpha_1 \sin\alpha_1 + k_2 \cos\alpha_2 \sin\alpha_2\right)\Delta X + \\ +\left(k_1 \sin^2\alpha_1 + k_2 \sin^2\alpha_2\right)\Delta Y = P_Y, \end{cases} \tag{3}$$

the determinant of which being

$$\Delta = \begin{vmatrix} k_1 \cos^2\alpha_1 + & k_1 \cos\alpha_1 \sin\alpha_1 + \\ k_2 \cos^2\alpha_2 & k_2 \cos\alpha_2 \sin\alpha_2 \\ k_1 \cos\alpha_1 \sin\alpha_1 + & k_1 \sin^2\alpha_1 + \\ k_2 \cos\alpha_2 \sin\alpha_2 & k_2 \sin^2\alpha_2 \end{vmatrix} = \tag{4}$$

$$k_1 k_2 \left(\sin\alpha_1 \cos\alpha_2 - \sin\alpha_2 \cos\alpha_1\right)^2 =$$

$$k_1 k_2 \sin^2\left(\alpha_1 - \alpha_2\right).$$

The compatibility is assured if $\Delta \neq 0$.

The condition $\Delta = 0$ implies $\alpha_2 = \alpha_1$ or $\alpha_2 = \alpha_1 + \pi$.

Let us consider the first case defined by $\alpha_2 = \alpha_1 = 0$. The system (3) transforms into

$$\begin{cases} \left(k_1 + k_2\right)\Delta X + 0 \cdot \Delta Y = P_X , \\ 0 \cdot \Delta X + 0 \cdot \Delta Y = P_Y . \end{cases} \tag{5}$$

If $P_Y = 0$, then it results

$$\left(k_1 + k_2\right)\Delta X = P_X , \tag{6}$$

wherefrom

$$\Delta X = \frac{P_X}{k_1 + k_2} , \tag{7}$$

the deviation $\Delta Y$ being undetermined.

If $P_Y \neq 0$ then the system (3) is a non-compatible one.

The second situation is characterized by $\alpha_2 = \alpha_1 + \pi = \pi$, while the system (3) takes the same form (5). The discussion is similar to that presented before.

### 3.2. Case of $n$ bars

The determinant of the system (2) reads

$$\Delta = \left(\sum_{i=1}^{n} k_i \cos^2\alpha_i\right)\left(\sum_{j=1}^{n} k_j \sin^2\alpha_j\right) - \\ -\left(\sum_{l=1}^{n} k_l \cos_l \sin_l\right)^2 , \tag{8}$$

wherefrom

$$\Delta = \sum_{i=1}^{n}\sum_{j=1}^{n} k_i k_j \sin^2\left(\alpha_i - \alpha_j\right). \tag{9}$$

The condition $\Delta = 0$ leads to $\alpha_i = \alpha_1$ or $\alpha_i = \alpha_1 + \pi$, $i = \overline{2, n}$.

In this situation we get

$$\begin{cases} \left(\sum_{i=1}^{n} k_i\right)\Delta X + 0 \cdot \Delta Y = P_X , \\ 0 \cdot \Delta X + \left(\sum_{i=1}^{n} k_i\right)\Delta Y = P_Y \end{cases} \tag{10}$$

and the discussion is similar to the above one.

## 4. CASE OF COMPATIBLE SYSTEM

### 4.1. General considerations

Denoting

$$a_{11} = \sum_{i=1}^{n} k_i \cos^2\alpha_i ,$$

$$a_{12} = a_{21} = \sum_{i=1}^{n} k_i \cos\alpha_i \sin\alpha_i , \tag{11}$$

$$a_{22} = \sum_{i=1}^{n} k_i \sin^2\alpha_i ,$$

the system (2) becomes

$$\begin{cases} a_{11}\Delta X + a_{12}\Delta Y = P_X \,, \\ a_{21}\Delta X + a_{22}\Delta Y = P_Y \,, \end{cases} \tag{12}$$

wherefrom

$$\Delta X = \frac{\begin{vmatrix} P_X & a_{12} \\ P_Y & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \; \Delta Y = \frac{\begin{vmatrix} a_{11} & P_X \\ a_{21} & P_Y \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \tag{13}$$

that is

$$\Delta X = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}} P_X - \frac{a_{12}}{a_{11}a_{22} - a_{12}a_{21}} P_Y \,, \tag{14}$$

$$\Delta Y = -\frac{a_{21}}{a_{11}a_{22} - a_{12}a_{21}} P_X + \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}} P_Y \,, \tag{15}$$

We denote

$$A = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \; B = -\frac{a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \\ C = \frac{a_{11}}{a_{11}a_{22} - a_{12}a_{21}}, \tag{16}$$

so that the expressions (14) and (15) become

$$\Delta X = A\,P_X + B\,P_Y \,, \tag{17}$$

$$\Delta Y = B\,P_X + C\,P_Y \,, \tag{18}$$

We have the following cases:

i) $\Delta > 0$, $a_{12} > 0$. It results $A > 0$, $B < 0$, $C > 0$;

ii) $\Delta > 0$, $a_{12} < 0$. One gets $A > 0$, $B > 0$, $C > 0$ ;

iii) $\Delta < 0$, $a_{12} > 0$. We have $A < 0$, $B > 0$, $C < 0$ ;

iv) $\Delta < 0$, $a_{12} < 0$. One obtains $A < 0$, $B < 0$, $C < 0$.

We consider that the excitations are harmonic, so that

$$P_X = P_X^0 + P_X^1 \cos(\omega_1 t + \varphi_1), \\ P_Y = P_Y^0 + P_Y^1 \cos(\omega_2 t + \varphi_2). \tag{19}$$

No matter which case is consider, the expressions (17) and (18) show that the solutions are combinations of harmonic functions

$$\Delta X = \Delta X^0 + \Delta X^1 \cos(\omega_1 t + \varphi_1) + \\ + \Delta X^2 \cos(\omega_2 t + \varphi_2), \tag{20}$$

$$\Delta Y = \Delta Y^0 + \Delta Y^1 \cos(\omega_1 t + \varphi_1) + \\ + \Delta Y^2 \cos(\omega_2 t + \varphi_2), \tag{21}$$

where $\Delta X^0$, $\Delta X^1$, $\Delta X^2$, $\Delta Y^0$, $\Delta Y^1$, $\Delta Y^2$ are constant parameters.

## 4.2. Ratio of pulsation is a rational number

In this situation there exists a rational positive number $r = \frac{p}{q}$, $p, q \in \mathbf{N}$, so that

$$\frac{\omega_1}{\omega_2} = \frac{p}{q}. \tag{22}$$

Denoting by $k$ the smallest common multiple of the number $p$ and $q$, the motions described by relations (20) and (21) are periodic of period equal to

$$T = \frac{2\pi}{k\omega}, \tag{23}$$

where

$$\omega = \frac{\omega_1}{p} = \frac{\omega_2}{q}. \tag{24}$$

The diagram looks like a closed Lissajous figure centered at the point of coordinates $\Delta X^0$ and $\Delta Y^0$, where

$$\Delta X^0 = A\,P_X^0 + B\,P_Y^0, \; \Delta Y^0 = B\,P_X^0 + C\,P_Y^0. \tag{25}$$

Depending on the values of $P_X^0$ and $P_Y^0$, and the cases presented before, one may obtain positive, negative or null values for $\Delta X^0$ and $\Delta Y^0$.

Zero values for $\Delta X^0$ and $\Delta Y^0$ are obtained if

$$\begin{cases} A\,P_X^0 + B\,P_Y^0 = 0\,, \\ B\,P_X^0 + C\,P_Y^0 = 0\,, \end{cases} \tag{26}$$

that is, $P_X^0 = 0$, $P_Y^0 = 0$.

Let us observe that if $AC - B^2 = 0$, then $\Delta = 0$ which is not our case.

### 4.3. Ratio of pulsations is an irrational number

In this situation the motion is not a periodic one and the diagram looks like an open Lissajous figure.

The figure is also centered at the point of coordinates $\Delta X^0$ and $\Delta Y^0$, the amplitudes of the motion being

$$A_X = \max\left[\Delta X^1 \cos(\omega_1 t + \varphi_1) + \Delta X^2 \cos(\omega_2 t + \varphi_2)\right], \quad (27)$$

$$A_Y = \max\left[\Delta Y^1 \cos(\omega_1 t + \varphi_1) + \Delta Y^2 \cos(\omega_2 t + \varphi_2)\right]. \quad (28)$$

## 5. CASE OF NON-COMPATIBLE SYSTEM

We will limit ourselves to the case of two bars. As we already saw we have two possibilities:

i) $\alpha_1 = \alpha_2 = 0$;

ii) $\alpha_1 = 0$, $\alpha_2 = \pi$.

### 5.1. First possibility

The situation is captured in Fig. 2 where the point $O$ moves into the new position $O'$.

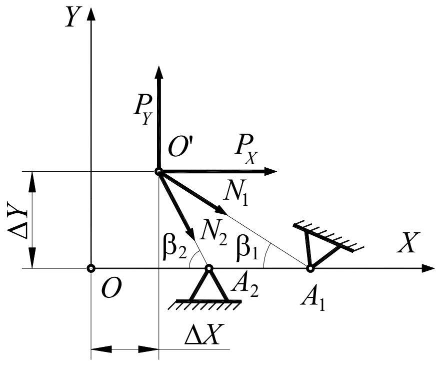


**Figure 2.** First possibility.

We have

$$\tan\beta_1 = \frac{\Delta Y}{l_1 - \Delta X}, \ \tan\beta_2 = \frac{\Delta Y}{l_2 - \Delta X}, \quad (29)$$

wherefrom

$$\cos\beta_i = \frac{1}{\sqrt{1+\tan^2\beta_i}} = \frac{l_i - \Delta X}{\sqrt{(l_i - \Delta X)^2 + (\Delta Y)^2}}, \ i = \overline{1,2}, \quad (30)$$

$$\sin\beta_i = \frac{\tan\beta_i}{\sqrt{1+\tan^2\beta_i}} = \frac{\Delta Y}{\sqrt{(l_i - \Delta X)^2 + (\Delta Y)^2}}, \ i = \overline{1,2}. \quad (31)$$

From the equations of equilibrium

$$P_X + N_1\cos\beta_1 + N_2\cos\beta_2 = 0, \quad P_Y - N_1\sin\beta_1 - N_2\sin\beta_2 = 0, \quad (32)$$

where

$$N_1 = k_1\left[\sqrt{(l_1 - \Delta X)^2 + (\Delta Y)^2} - l_1\right], \quad N_2 = k_2\left[\sqrt{(l_2 - \Delta X)^2 + (\Delta Y)^2} - l_2\right], \quad (33)$$

one obtains a system of two non-linear equations with two unknowns ($\Delta X$ and $\Delta Y$)

$$-k_1\left[\sqrt{(l_1 - \Delta X)^2 + (\Delta Y)^2} - l_1\right]\frac{l_1 - \Delta X}{\sqrt{(l_1 - \Delta X)^2 + (\Delta Y)^2}} - k_2\left[\sqrt{(l_2 - \Delta X)^2 + (\Delta Y)^2} - l_2\right]\frac{l_2 - \Delta X}{\sqrt{(l_2 - \Delta X)^2 + (\Delta Y)^2}} = P_X, \quad (34)$$

$$-k_1\left[\sqrt{(l_1 - \Delta X)^2 + (\Delta Y)^2} - l_1\right]\frac{\Delta Y}{\sqrt{(l_1 - \Delta X)^2 + (\Delta Y)^2}} + k_2\left[\sqrt{(l_2 - \Delta X)^2 + (\Delta Y)^2} - l_2\right]\frac{\Delta Y}{\sqrt{(l_2 - \Delta X)^2 + (\Delta Y)^2}} = P_Y. \quad (35)$$

Denoting

$$A = \sqrt{(l_1 - \Delta X)^2 + (\Delta Y)^2}, \quad B = \sqrt{(l_2 - \Delta X)^2 + (\Delta Y)^2}, \quad (36)$$

by $f_1(\Delta X, \Delta Y)$ and $f_2(\Delta X, \Delta Y)$ the left-hand terms of the equations (34) and (35), knowing that

$$\frac{\partial A}{\partial \Delta X} = A_X = \frac{-(l_1 - \Delta X)}{A}, \ \frac{\partial A}{\partial \Delta Y} = A_Y = \frac{\Delta Y}{A}, \quad (37)$$

$$\frac{\partial B}{\partial \Delta X} = B_X = \frac{-(l_2 - \Delta X)}{B}, \ \frac{\partial B}{\partial \Delta Y} = B_Y = \frac{\Delta Y}{B}, \quad (38)$$

it results

$$\frac{\partial f_1}{\partial \Delta X} = -k_1 A_X \frac{l_1 - \Delta X}{A} -$$
$$k_1 (A - l_1) \frac{-A - (l_1 - \Delta X) A_X}{A^2} - \tag{39}$$
$$k_2 B_X \frac{l_2 - \Delta X}{B} - k_2 (B - l_2) \frac{-B - (l_2 - \Delta X) B_X}{B^2},$$

$$\frac{\partial f_1}{\partial \Delta Y} = -k_1 A_Y \frac{l_1 - \Delta X}{A} -$$
$$-k_1 (A - l_1) \frac{-(l_1 - \Delta X) A_Y}{A^2} - \tag{40}$$
$$-k_2 B_Y \frac{l_2 - \Delta X}{B} - k_2 (B - l_2) \frac{-(l_2 - \Delta X) B_Y}{B^2},$$

$$\frac{\partial f_2}{\partial \Delta X} = k_1 A_X \frac{\Delta Y}{A} +$$
$$+k_1 (A - l_1) \frac{-\Delta Y\, A_X}{A^2} + \tag{41}$$
$$+k_2 B_X \frac{\Delta Y}{B} + k_2 (B - l_2) \frac{-\Delta Y\, B_X}{B^2},$$

$$\frac{\partial f_2}{\partial \Delta Y} = k_1 A_Y \frac{\Delta Y}{B} +$$
$$+k_1 (A - l_1) \frac{-\Delta Y\, A_Y}{A^2} + \tag{42}$$
$$+k_2 B_Y \frac{\Delta Y}{B} + k_2 (B - l_2) \frac{-\Delta Y\, B_Y}{B^2}.$$

The variations $\Delta(\Delta X)$ and $\Delta(\Delta Y)$ are obtained from the system

$$\begin{bmatrix} \frac{\partial f_1}{\partial \Delta X} & \frac{\partial f_1}{\partial \Delta Y} \\ \frac{\partial f_2}{\partial \Delta X} & \frac{\partial f_2}{\partial \Delta Y} \end{bmatrix} \begin{bmatrix} \Delta(\Delta X) \\ \Delta(\Delta Y) \end{bmatrix} = \begin{bmatrix} -f_1(\Delta X, \Delta Y) + P_X \\ -f_2(\Delta X, \Delta Y) + P_Y \end{bmatrix}. \tag{43}$$

As a particular case we consider that $k_1 = k_2$, $l_1 = l_2 = l$, $P_X = P_Y = P$. The equations (34) and (35) lead to the system

$$-2k\left[\sqrt{(l - \Delta X)^2 + (\Delta Y)^2} - l\right]$$
$$\frac{l - \Delta X}{\sqrt{(l - \Delta X)^2 + (\Delta Y)^2}} = P,$$
$$2k\left[\sqrt{(l - \Delta X)^2 + (\Delta Y)^2} - l\right] \tag{44}$$
$$\frac{\Delta Y}{\sqrt{(l - \Delta X)^2 + (\Delta Y)^2}} = P,$$

wherefrom

$$l - \Delta X = -\Delta Y. \tag{45}$$

Replacing relation (45) in the first expression (44), one gets

$$-2k\left[|l - \Delta X|\sqrt{2} - l\right] \frac{l - \Delta X}{|l - \Delta X|\sqrt{2}} = P, \tag{46}$$

that is

$$l - \Delta X - \frac{l(l - \Delta X)}{|l - \Delta X|\sqrt{2}} = -\frac{P}{2k}, \tag{47}$$

or, equivalently,

$$l - \Delta X = -\frac{P}{2k} + \frac{l}{\sqrt{2}}, \tag{48}$$

that is

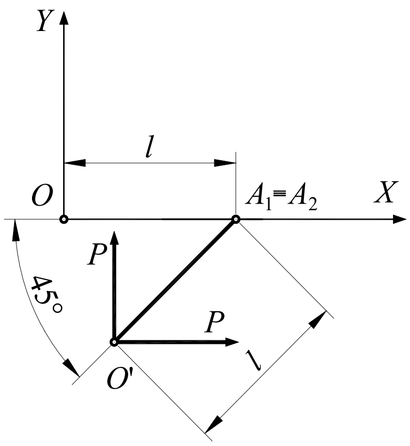


**Figure 3.** Equilibrium position in the first case.

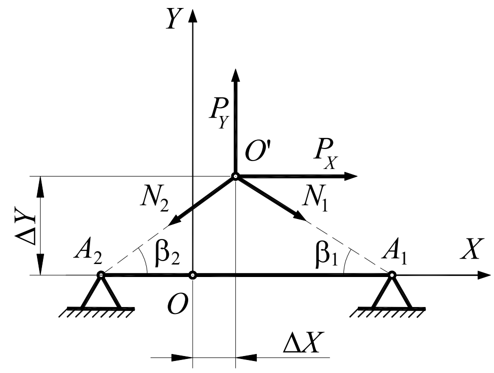


**Figure 4.** Second possibility.

$$\Delta X = l - \frac{l}{\sqrt{2}} + \frac{P}{2k} = \frac{l(\sqrt{2} - 1)}{\sqrt{2}} + \frac{P}{2k},$$
$$\Delta Y = \frac{P}{2k} - \frac{l}{\sqrt{2}} \tag{49}$$

and

$$\tan\beta_1 = \tan\beta_2 = \tan\beta = \frac{\Delta Y}{l - \Delta X} = -1,$$
$$\beta = -\frac{\pi}{4}. \tag{50}$$

The equilibrium position is captured in Fig. 3.

### 5.2. Second possibility

This situation is presented in Fig. 4. It results

$$\tan\beta_1 = \frac{\Delta Y}{l_1 - \Delta X},\ \tan\beta_2 = \frac{\Delta Y}{l_2 - \Delta X}, \quad (51)$$

$$\cos\beta_1 = \frac{1}{\sqrt{1+\tan^2\beta_1}} = \frac{l_1 - \Delta X}{\sqrt{(l_1-\Delta X)^2 + (\Delta Y)^2}},$$
$$\cos\beta_2 = \frac{1}{\sqrt{1+\tan^2\beta_2}} = \frac{l_2 + \Delta X}{\sqrt{(l_2-\Delta X)^2 + (\Delta Y)^2}}, \quad (52)$$

$$\sin\beta_1 = \frac{\tan\beta_1}{\sqrt{1+\tan^2\beta_1}} = \frac{\Delta Y}{\sqrt{(l_1-\Delta X)^2 + (\Delta Y)^2}},$$
$$\sin\beta_2 = \frac{\tan\beta_2}{\sqrt{1+\tan^2\beta_2}} = \frac{\Delta Y}{\sqrt{(l_2-\Delta X)^2 + (\Delta Y)^2}}, \quad (53)$$

The equations of equilibrium lead to

$$P_X + N_1\cos\beta_1 - N_2\cos\beta_2 = 0,$$
$$P_Y - N_1\sin\beta_1 - N_2\sin\beta_2 = 0, \quad (54)$$

wherefrom one gets the following system of two non-linear equations with two unknowns ($\Delta X$ and $\Delta Y$),

$$-k_1\left[\sqrt{(l_1-\Delta X)^2+(\Delta Y)^2} - l_1\right] \frac{l_1-\Delta X}{\sqrt{(l_1-\Delta X)^2+(\Delta Y)^2}} + k_2\left[\sqrt{(l_2+\Delta X)^2+(\Delta Y)^2} - l_2\right] \frac{l_2+\Delta X}{\sqrt{(l_2+\Delta X)^2+(\Delta Y)^2}} = -P_X, \quad (55)$$

$$k_1\left[\sqrt{(l_1-\Delta X)^2+(\Delta Y)^2} - l_1\right] \frac{\Delta Y}{\sqrt{(l_1-\Delta X)^2+(\Delta Y)^2}} + k_2\left[\sqrt{(l_2+\Delta X)^2+(\Delta Y)^2} - l_2\right] \frac{\Delta Y}{\sqrt{(l_2+\Delta X)^2+(\Delta Y)^2}} = P_Y. \quad (56)$$

As a particular case we consider that, $l_1 = l_2 = l$, $\Delta X = 0$, $k_1 = k_2 = k$, $\alpha_1 = 0$, $\alpha_2 = \pi$. It results the system

$$-k\left[\sqrt{l^2+(\Delta Y)^2} - l\right]\frac{l}{\sqrt{l^2+(\Delta Y)^2}} + k\left[\sqrt{l^2+(\Delta Y)^2} - l\right]\frac{l}{\sqrt{l^2+(\Delta Y)^2}} = P_X, \quad (57)$$

$$k\left[\sqrt{l^2+(\Delta Y)^2} - l\right]\frac{\Delta Y}{\sqrt{l^2+(\Delta Y)^2}} + k\left[\sqrt{l^2+(\Delta Y)^2} - l\right]\frac{\Delta Y}{\sqrt{l^2+(\Delta Y)^2}} = P_Y, \quad (58)$$

wherefrom $P_X = 0$ and

$$\frac{\sqrt{l^2+(\Delta Y)^2} - l}{\sqrt{l^2+(\Delta Y)^2}}\Delta Y = \frac{P_Y}{2k}. \quad (59)$$

It successively results

$$-\frac{l}{\sqrt{l^2+(\Delta Y)^2}}\Delta Y = \frac{P_Y}{2k} - \Delta Y, \quad (60)$$

$$\frac{l^2(\Delta Y)^2}{l^2+(\Delta Y)^2} = \left(\frac{P_Y}{2k}\right)^2 + (\Delta Y)^2 - \frac{P_Y}{k}\Delta Y, \quad (61)$$

$$(\Delta Y)^4 - \frac{P_Y}{k}(\Delta Y)^3 + \left(\frac{P_Y}{2k}\right)^2(\Delta Y)^2 - \frac{P_Y}{2k}l^2\Delta Y + \left(\frac{P_Y}{2k}\right)^2 l^2 = 0, \quad (62)$$

that is a four-degree algebraic equation.

If $\frac{P_Y}{k} < l$, then one may observe that the equation (61) has at least one root between 0 and $\frac{P_Y}{k}$.

## 6. NUMERICAL RESULTS

We consider the situation described in Fig. 5, where $l_1 = 0.1\ \text{m}$, $l_2 = 0.25\ \text{m}$, $l_3 = 0.2\ \text{m}$, $E_1 = E_2 = E_3 = 2\cdot 10^9\ \text{Pa}$, $A_1 = A_2 = A_3 = 4\cdot 10^{-6}\ \text{m}^2$, $\beta_1 = 45^0$, $\beta_2 = 30^0$.

It results $k_1 = 80000\ \text{N/m}$, $k_2 = 32000\ \text{N/m}$, $k_3 = 40000\ \text{N/m}$, $\alpha_1 = 45^0$, $\alpha_2 = 150^0$, $\alpha_3 = 270^0$, $\cos\alpha_1 = \frac{\sqrt{2}}{2}$, $\sin\alpha_1 = \frac{\sqrt{2}}{2}$, $\cos\alpha_2 = -\frac{\sqrt{3}}{2}$, $\sin\alpha_2 = \frac{1}{2}$, $\cos\alpha_3 = 0$, $\sin\alpha_3 = -1$, $\cos^2\alpha_1 = \frac{1}{2}$,

$\sin^2\alpha_1 = \frac{1}{2}$, $\cos\alpha_1 \sin\alpha_1 = \frac{1}{2}$, $\cos^2\alpha_2 = \frac{3}{4}$, $\sin^2\alpha_2 = \frac{1}{4}$, $\cos\alpha_2 \sin\alpha_2 = -\frac{\sqrt{3}}{4}$, $\cos^2\alpha_3 = 0$, $\sin^2\alpha_3 = 1$, $\cos\alpha_3 \sin\alpha_3 = 0$,

$$\sum_{i=1}^{3} k_i \cos^2\alpha_i = 64000 \text{ N/m},$$

$$\sum_{i=1}^{3} k_i \cos\alpha_i \sin\alpha_i = 26143.6 \text{ N/m},$$

$$\sum_{i=1}^{3} k_i \sin^2\alpha_i = 88000 \text{ N/m}.$$

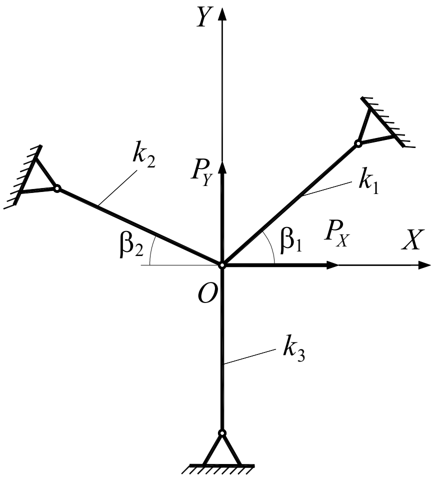


**Figure 5.** Numerical application.

One obtains the system

$$\begin{cases} 64000\,\Delta X + 26143.6\,\Delta Y = P_X, \\ 26143.6\,\Delta X + 88000\,\Delta Y = P_Y. \end{cases} \tag{63}$$

We consider the following four variants:

i) $P_X = 200$ N, $P_Y = 150$ N;

ii) $P_X = 200 + 50\cos(3t)$, $P_Y = 150 + 60\cos(3t)$;

iii) in which $P_X = 200 + 50\cos(3t) + 10\cos(10t)$, $P_Y = 150 + 60\cos(3t) + 15\cos(10t)$;

iv) in which $P_X = 200 + 50\cos(3t) + 10\cos(2\pi t)$, $P_Y = 150 + 60\cos(3t) + 15\cos(2\pi t)$.

In all cases $t \in [0; 20]$ s.

For the first variant one obtains the constant deviations $\Delta X = 0.002764$ m, $\Delta Y = 0.000883$ m.

The graphics for the rest variants are given in the next figures (Fig. 6… Fig. 8).

For the first possibility detailed in paragraph 5.1 we select the values: $\frac{P}{2k} = 0.005 + 0.0025\cos(3t)$, $l = 0.2$ m, $t_{max} = 5.0$ s, $\Delta t = 10^{-3}$ s, while for the second possibility described in the same paragraph we have chosen: $\frac{P}{k} = 0.01 + 0.005\cos(4t)$, $l = 0.2$ m, $t_{max} = 5.0$ s, and $\Delta t = 10^{-3}$ s. The graphics are captured in Fig. 9 and Fig. 10.

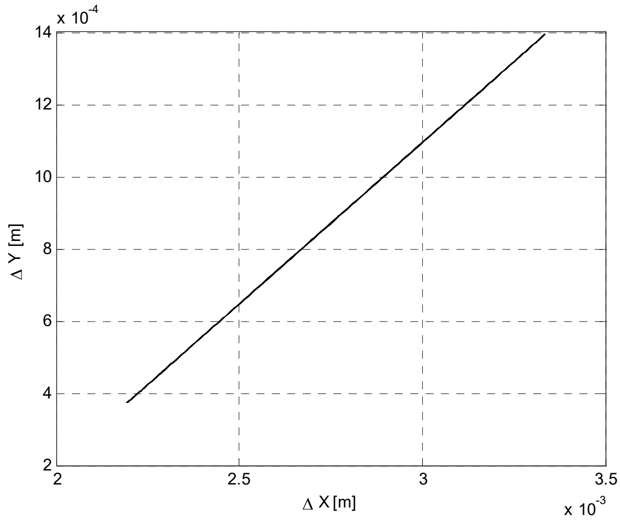


**Figure 6.** Diagram $\Delta Y = \Delta Y(\Delta X)$, second variant.

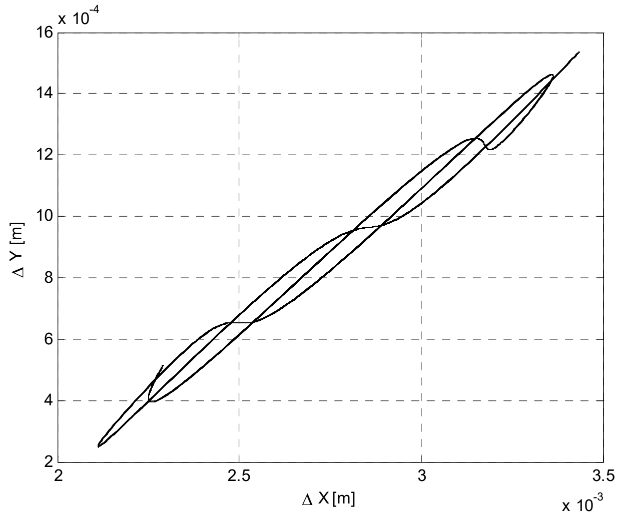


**Figure 7.** Diagram $\Delta Y = \Delta Y(\Delta X)$, third variant.

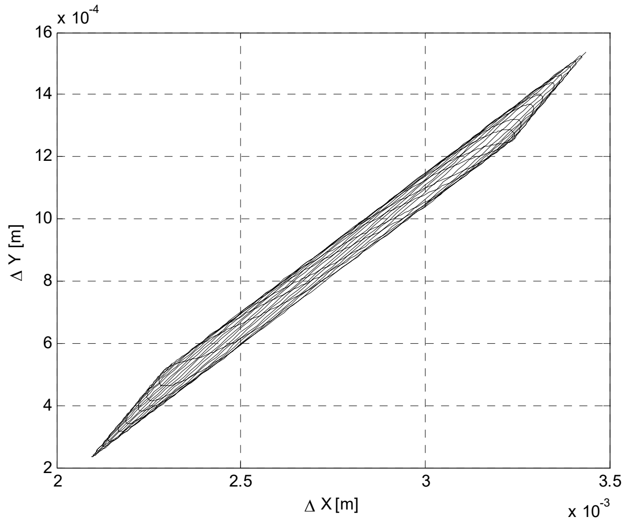


**Figure 8.** Diagram $\Delta Y = \Delta Y(\Delta X)$, fourth variant.

Analyzing the previous diagrams one may state that in the case of the second variant the diagram consists in a segment of straight line because one may eliminate $\cos(3t)$ in the expressions of $P_X(t)$ and $P_Y(t)$ resulting a linear relation between $P_X$ and $P_Y$. Taking into account the periodicity of the function $\cos(3t)$ one may state that the segment is described by infinite times.

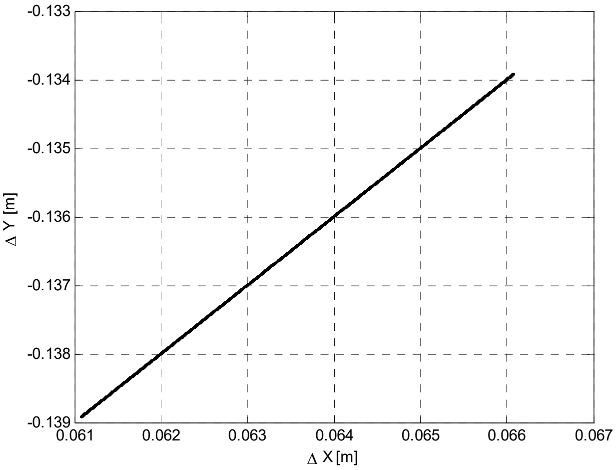


**Figure 9.** Diagram $\Delta Y = \Delta Y(\Delta X)$, first possibility.

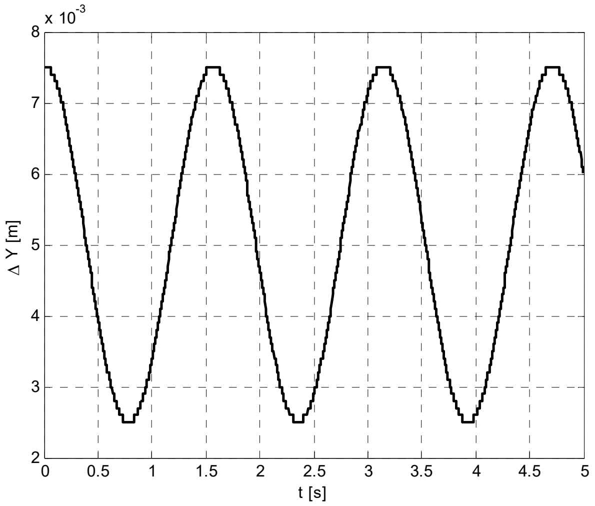


**Figure 10.** Diagram $\Delta Y = \Delta Y(t)$, second possibility.

In the case of the third variant the ratio $\frac{\omega_1}{\omega_2}$ is a rational number resulting that the curve in Fig. 7 is a closed and periodic one, the period being $T = 2\pi$ s.

In Fig. 8 we have the case in which the ratio $\frac{\omega_1}{\omega_2}$ is an irrational number and the diagram is not a periodic one.

Figure 9 is a simply consequence of the linearity of the equations (49). In this situation the diagram is a segment of a straight line and, taking into account the periodicity of the function $\cos(3t)$, it results that it is also periodic.

For the Figure 10, we have chosen only the solution $\Delta Y$ which is situated between 0 and $\frac{P_Y}{k}$. It is easy to observe that, in the hypotheses assumed in this case, there exists at least another solution between $\frac{P_Y}{k}$ and infinite. The diagram is periodic with the period $T = \frac{2\pi}{4} = \frac{\pi}{2}$ s.

## 7. CONCLUSIONS

In this paper we discussed the compatibility situations for a planar system of bars cylindrically jointed. It resulted that the problem can be solved for small deformations if and only if the bars define at least two independent directions in plane. For the case in which this condition does not hold true we developed the general theory from which one may determine the two deviations, $\Delta X$ and $\Delta Y$.

For both situations we presented complete solved numerical applications. Even when the ratio of the pulsations is an irrational number one may obtain periodic diagrams for the deviations, but only in very particular systems.

The case of spatial systems will be discussed in our future work.

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