On the Dynamics of the Constrained Rigid Solid Acted by Conservative Forces. Part I: Theory

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Abstract: - This paper analyses the dynamics of a rigid solid, the main goal being the obtaining of the matrix differential equation of motion. The rigid solid is a constrained one and it is acted by conservative forces. The approach is a multibody one and it is valid for the general case. We also deal with the planar motion of the rigid solid. As a particular case we discuss the equilibrium of the rigid solid. Some comments are also presented.

Keywords: - multibody approach, conservative forces, equation of motion, equilibrium

1. INTRODUCTION

Many authors present the multibody theory [1 – 5] and, after the obtaining of the matrix differential equation of motion, they discuss simple examples. Usually, these examples can be solved by using the classical theorems of mechanics (theorem of momentum and theorem of moment of momentum). The general solution of the matrix differential equation of motion is usually avoided by the reason that the inertial matrix is not always invertible. A method by which a rotational schema is transformed in another one in order to avoid the singularity of the inertial matrix is described in [6]. For the case of conservative forces authors prefer the equations of motion obtained from the second order Lagrange equations. They also consider simple examples. The particular case of planar systems are treated in [4], while some cases referring the robots are discussed in [5]. Generally speaking, the systems are resulted from the consideration of some particular mechanisms, the solution obtained from the multibody approach being compared to that obtained by classical methods.

Usually, the constraints are considered to be bilateral, but there are also references that deal with unilateral constraints [7]. The great challenge in this case is that the number of constraints is variable so that the calculation program must consider all possible situations.

Some aspects concerning the simulation of the multibody systems are presented in [1 – 12]. The authors discuss the major problems that may appear in the simulation, but no calculation program is presented. This disadvantage has as its main cause the length of these calculation programs, so that the books would become huge. The problems which include both multibody systems and their nonsmooth behavior are discussed in [13, 14].

The general case of conservative forces is discussed in [15], while the equation of motion is obtained from the second order Lagrange equations. The situation when the rigid solid is constrained is not discussed. The particular case of linear springs is treated only for the determination of the equilibrium position.

In this paper we obtain the matrix differential equation of motion in a multibody form for the situation of many conservative forces acting upon the rigid solid. The equilibrium positions are also obtained.

2. PROBLEM FORMULATION

Let us consider a fixed frame O_0XYZ (Fig. 1), the rigid solid and the mobile reference frame $Oxyz$ rigidly connected to the body. We denote by C the center of weight of the rigid solid. At the point A the rigid solid is acted by the force $\mathbf{F}$ which is conservative, that is, there exists the potential V so that

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}\mathbf{i}_0 + \frac{\partial V}{\partial y}\mathbf{j}_0 + \frac{\partial V}{\partial z}\mathbf{k}_0\right), \quad (1)$$

where $\mathbf{i}_0$, $\mathbf{j}_0$ and $\mathbf{k}_0$ are the unit vectors of the fixed axes O_0X, O_0Y and O_0Z.

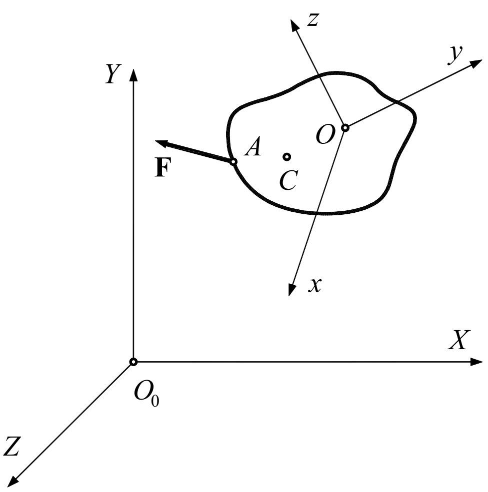

Figure 1. Rigid solid acted by conservative forces.

The unit vectors of the mobile axes are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

One knows the mass m of the rigid solid, the moments of inertia J_x, J_y, J_z, J_{xy}, J_{yz}, J_{zx} of the rigid solid relative to the mobile axes, the coordinates x_C, y_C, z_C of the center of weight relative to the mobile frame, and the coordinates x_A, y_A, z_A of the point A relative to the mobile axes.

One asks for the determination of the motion of the rigid solid.

3. NOTATIONS

We consider that the rotational schema is a Bryan one given by the order x, y, z.

We denote [15]:

– the angles of rotation ψ, θ, φ about the axes x, y, z to which correspond the matrices

$$[\boldsymbol{\psi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix},$$

$$[\boldsymbol{\theta}] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad (2)$$

$$[\boldsymbol{\varphi}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

– the matrix of rotation

$$[\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}] =$$
$$\begin{bmatrix} c\theta c\varphi & -c\theta s\varphi & s\theta \\ s\psi s\theta c\varphi + c\psi s\varphi & -s\psi s\theta s\varphi + c\psi c\varphi & -s\psi c\theta \\ c\psi s\theta c\varphi + s\psi s\varphi & c\psi s\theta s\varphi + s\psi c\varphi & c\psi c\theta \end{bmatrix}; \quad (3)$$

– the parameters X_O, Y_O, Z_O, ψ, θ, φ that give the position of the rigid solid for which correspond the column matrices

$$\{\mathbf{s}\} = [X_O \quad Y_O \quad Z_O]^{\mathrm{T}},\ \{\boldsymbol{\beta}\} = [\psi \quad \theta \quad \varphi]^{\mathrm{T}}, \quad (4)$$

$$\{\mathbf{q}\} = [X_O \quad Y_O \quad Z_O \quad \psi \quad \theta \quad \varphi]^{\mathrm{T}}; \quad (5)$$

– the matrix $[\mathbf{S}]$ given by

$$[\mathbf{S}] = \begin{bmatrix} 0 & -mz_C & my_C \\ mz_C & 0 & -mx_C \\ -my_C & mx_C & 0 \end{bmatrix}; \quad (6)$$

– the matrix of the moments of inertia

$$[\mathbf{J}_O] = \begin{bmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{bmatrix}; \quad (7)$$

– the matrix $[\mathbf{m}]$ given by

$$[\mathbf{m}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}; \quad (8)$$

– the column matrices

$$\{\mathbf{u}_\psi\} = [1 \quad 0 \quad 0]^T,\ \{\mathbf{u}_\theta\} = [0 \quad 1 \quad 0]^T,\ \{\mathbf{u}_\varphi\} = [0 \quad 0 \quad 1]^T; \tag{9}$$

– the matrix $[\mathbf{Q}]$ given by

$$[\mathbf{Q}] = [\boldsymbol{\varphi}]^T \left[[\boldsymbol{\theta}]^T \{\mathbf{u}_\psi\} \quad \{\mathbf{u}_\theta\} \quad \{\mathbf{u}_\varphi\} \right]; \tag{10}$$

– the square matrices

$$[\mathbf{U}_\psi] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ [\mathbf{U}_\theta] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ [\mathbf{U}_\varphi] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{11}$$

– the matrices

$$[\boldsymbol{\psi}_p] = [\mathbf{U}_\psi][\boldsymbol{\psi}],\ [\boldsymbol{\theta}_p] = [\mathbf{U}_\theta][\boldsymbol{\theta}],\ [\boldsymbol{\varphi}_p] = [\mathbf{U}_\varphi][\boldsymbol{\varphi}]; \tag{12}$$

– the partial derivatives of the rotational matrix

$$[\mathbf{A}_\psi] = [\boldsymbol{\psi}_p][\boldsymbol{\theta}][\boldsymbol{\varphi}],\ [\mathbf{A}_\theta] = [\boldsymbol{\psi}][\boldsymbol{\theta}_p][\boldsymbol{\varphi}],\ [\mathbf{A}_\varphi] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}_p]; \tag{13}$$

– the partial derivatives of the matrix $[\mathbf{Q}]$

$$[\mathbf{Q}_\varphi] = [\boldsymbol{\varphi}_p]^T \left[[\boldsymbol{\theta}]^T \{\mathbf{u}_\psi\} \quad \{\mathbf{u}_\theta\} \quad \{\mathbf{u}_\varphi\} \right],\ [\mathbf{Q}_\theta] = [\boldsymbol{\varphi}]^T \left[[\boldsymbol{\theta}_p]^T \{\mathbf{u}_\psi\} \quad \{\mathbf{0}\} \quad \{\mathbf{0}\} \right]; \tag{14}$$

– the derivative of the matrix $[\mathbf{A}]$ with respect to time

$$[\dot{\mathbf{A}}] = \dot{\psi}[\mathbf{A}_\psi] + \dot{\theta}[\mathbf{A}_\theta] + \dot{\varphi}[\mathbf{A}_\varphi]; \tag{15}$$

– the derivative of the matrix $[\mathbf{Q}]$ with respect to time

$$[\dot{\mathbf{Q}}] = \dot{\varphi}[\mathbf{Q}_\varphi] + \dot{\theta}[\mathbf{Q}_\theta]; \tag{16}$$

– the matrices $\{\boldsymbol{\omega}\}$ and $[\boldsymbol{\omega}]$ corresponding to the angular velocity

$$\{\boldsymbol{\omega}\} = [\mathbf{Q}]\{\dot{\boldsymbol{\beta}}\} = [\omega_x \quad \omega_y \quad \omega_z]^T,\ \{\boldsymbol{\omega}\} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}; \tag{17}$$

– the matrix of inertia

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{m}] & [\mathbf{A}][\mathbf{S}]^T[\mathbf{Q}] \\ [\mathbf{Q}]^T[\mathbf{S}][\mathbf{A}]^T & [\mathbf{Q}]^T[\mathbf{J}_O][\mathbf{Q}] \end{bmatrix}; \tag{18}$$

– the matrices $\{\widetilde{\mathbf{F}}_\mathbf{s}\}$, $\{\widetilde{\mathbf{F}}_\boldsymbol{\beta}\}$ given by

$$\{\widetilde{\mathbf{F}}_\mathbf{s}\} = -\left[[\mathbf{A}][\mathbf{S}]^T[\dot{\mathbf{Q}}] + [\dot{\mathbf{A}}][\mathbf{S}]^T[\mathbf{Q}]\right]\{\dot{\boldsymbol{\beta}}\}, \tag{19}$$

$$\{\widetilde{\mathbf{F}}_\boldsymbol{\beta}\} = -\left[[\mathbf{Q}]^T[\mathbf{J}_0][\dot{\mathbf{Q}}] + [\mathbf{Q}]^T[\boldsymbol{\omega}][\mathbf{J}_0][\mathbf{Q}]\right]\{\dot{\boldsymbol{\beta}}\}; \tag{20}$$

– the matrix of constraints $[\mathbf{B}]$;

– the conservative force

$$\{\mathbf{F}\} = [F_X \quad F_Y \quad F_Z]^T, \tag{21}$$

where $F_X = F_X(X_A, Y_A, Z_A)$, $F_Y = F_Y(X_A, Y_A, Z_A)$, $F_Z = F_Z(X_A, Y_A, Z_A)$.

– the matrices $[\mathbf{r}_A]$ and $\{\mathbf{r}_A\}$ given by

$$[\mathbf{r}_A] = \begin{bmatrix} 0 & -z_A & y_A \\ z_A & 0 & -x_A \\ -y_A & x_A & 0 \end{bmatrix},\ \{\mathbf{r}_A\} = [x_A \quad y_A \quad z_A]^T. \tag{22}$$

4. MATRIX DIFFERENTIAL EQUATION OF MOTION

Denoting with X_A, Y_A, Z_A the coordinates of the point A relative to the fixed reference frame, one way write the matrix relation

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} = \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix} + [\mathbf{A}]\{\mathbf{r}_A\}. \tag{23}$$

Since

$$\frac{\partial X_A}{\partial X_O} = 1,\ \frac{\partial Y_A}{\partial Y_O} = 1,\ \frac{\partial Z_A}{\partial Z_O} = 1, \tag{24}$$

$$\frac{\partial V}{\partial X_O}=\frac{\partial V}{\partial X_A}\frac{\partial X_A}{\partial X_O},\ \frac{\partial V}{\partial Y_O}=\frac{\partial V}{\partial Y_A}\frac{\partial Y_A}{\partial Y_O},\ \frac{\partial V}{\partial Z_O}=\frac{\partial V}{\partial Z_A}\frac{\partial Z_A}{\partial Z_O},\tag{25}$$

it results the generalized force that acts upon the rigid solid at the point A

$$F_{X_O}=-\frac{\partial V}{\partial X_O}=-\frac{\partial V}{\partial X_A},\quad F_{Y_O}=-\frac{\partial V}{\partial Y_O}=-\frac{\partial V}{\partial Y_A},\quad F_{Z_O}=-\frac{\partial V}{\partial Z_O}=-\frac{\partial V}{\partial Z_A}\frac{\partial Z_A}{\partial Z_O}.\tag{26}$$

We also have

$$\frac{\partial V}{\partial \psi}=\frac{\partial V}{\partial X_A}\frac{\partial X_A}{\partial \psi}+\frac{\partial V}{\partial Y_A}\frac{\partial Y_A}{\partial \psi}+\frac{\partial V}{\partial Z_A}\frac{\partial Z_A}{\partial \psi},\quad \frac{\partial V}{\partial \theta}=\frac{\partial V}{\partial X_A}\frac{\partial X_A}{\partial \theta}+\frac{\partial V}{\partial Y_A}\frac{\partial Y_A}{\partial \theta}+\frac{\partial V}{\partial Z_A}\frac{\partial Z_A}{\partial \theta},\quad \frac{\partial V}{\partial \varphi}=\frac{\partial V}{\partial X_A}\frac{\partial X_A}{\partial \varphi}+\frac{\partial V}{\partial Y_A}\frac{\partial Y_A}{\partial \varphi}+\frac{\partial V}{\partial Z_A}\frac{\partial Z_A}{\partial \varphi}\tag{27}$$

and one gets

$$\frac{\partial V}{\partial \psi}=-\{\mathbf{r}_A\}^{\mathrm{T}}[\mathbf{A}_\psi]^{\mathrm{T}}\begin{bmatrix}F_{X_O}\\F_{Y_O}\\F_{Z_O}\end{bmatrix},\quad \frac{\partial V}{\partial \theta}=-\{\mathbf{r}_A\}^{\mathrm{T}}[\mathbf{A}_\theta]^{\mathrm{T}}\begin{bmatrix}F_{X_O}\\F_{Y_O}\\F_{Z_O}\end{bmatrix},\quad \frac{\partial V}{\partial \varphi}=-\{\mathbf{r}_A\}^{\mathrm{T}}[\mathbf{A}_\varphi]^{\mathrm{T}}\begin{bmatrix}F_{X_O}\\F_{Y_O}\\F_{Z_O}\end{bmatrix};\tag{28}$$

consequently

$$F_\psi=-\frac{\partial V}{\partial \psi},\ F_\theta=-\frac{\partial V}{\partial \theta},\ F_\varphi=-\frac{\partial V}{\partial \varphi}.\tag{29}$$

The matrix $\{\mathbf{F}\}$ takes the form

$$\{\mathbf{F}\}=-\begin{bmatrix}\frac{\partial V}{\partial X_O} & \frac{\partial V}{\partial Y_O} & \frac{\partial V}{\partial Z_O} & \frac{\partial V}{\partial \psi} & \frac{\partial V}{\partial \theta} & \frac{\partial V}{\partial \varphi}\end{bmatrix}^{\mathrm{T}}\tag{30}$$

or, equivalently,

$$\{\mathbf{F_q}\}=\{\mathbf{F}\}=\begin{bmatrix}[\mathbf{I}]\\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}F_{X_O}\\F_{Y_O}\\F_{Z_O}\end{bmatrix}.\tag{31}$$

According to [15], the matrix differential equation of motion reads

$$\begin{bmatrix}[\mathbf{M}] & -[\mathbf{B}]^{\mathrm{T}}\\ [\mathbf{B}] & [\mathbf{0}]\end{bmatrix}\begin{bmatrix}\{\ddot{\mathbf{q}}\}\\ \{\ddot{\boldsymbol{\lambda}}\}\end{bmatrix}=\begin{bmatrix}\{\mathbf{F_q}\}+\{\widetilde{\mathbf{F}}_\mathbf{q}\}\\ \{\dot{\mathbf{C}}\}-[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}\end{bmatrix},\tag{32}$$

where

$$\{\mathbf{C}\}=[\mathbf{B}]\{\dot{\mathbf{q}}\}\tag{33}$$

$$\{\widetilde{\mathbf{F}}_\mathbf{q}\}=\begin{bmatrix}\{\widetilde{\mathbf{F}}_\mathbf{s}\}^{\mathrm{T}} & \{\widetilde{\mathbf{F}}_\boldsymbol{\beta}\}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\tag{34}$$

and $\{\boldsymbol{\lambda}\}$ is the column matrix of Lagrange multipliers.

If the rigid solid is acted by many forces $\mathbf{F}_1$, $\mathbf{F}_2$, …, $\mathbf{F}_l$, at the points A_1, A_2, …, A_l, then one has to calculate the matrices $\{\mathbf{F_q}\}_1$, $\{\mathbf{F_q}\}_2$, …, $\{\mathbf{F_q}\}_l$ corresponding to each force, using formulae similar to equation (31) and to sum these column matrices in order to obtain the column matrix $\{\mathbf{F_q}\}$. The calculation may now continue obtaining the matrix differential equation of motion.

5. PARTICULAR CASES

If the origin of the mobile reference system is takes at the center of weight of the rigid solid, then the matrix $[\mathbf{S}]=[\mathbf{0}]$; consequently

$$[\mathbf{M}]=\begin{bmatrix}[\mathbf{m}] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}]\end{bmatrix},\tag{35}$$

$$\{\mathbf{F_s}\}=\{\mathbf{0}\}=[0\quad 0\quad 0]^{\mathrm{T}}.\tag{36}$$

For the Bryan rotational schema we have

$$[\mathbf{Q}]=\begin{bmatrix}\cos\varphi\cos\theta & \sin\varphi & 0\\ -\sin\varphi\cos\theta & \cos\varphi & 0\\ \sin\theta & 0 & 1\end{bmatrix},\tag{37}$$

$$\{\boldsymbol{\omega}\}=\begin{bmatrix}\dot\psi\cos\varphi\cos\theta+\dot\theta\sin\varphi\\ -\dot\psi\sin\varphi\cos\theta+\dot\theta\cos\varphi\\ \dot\psi\sin\theta+\dot\varphi\end{bmatrix}.\tag{38}$$

If the axes of the mobile frame are chosen as the principal central axes of inertia, then

$$[\mathbf{J}_C]=\begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}, \tag{39}$$

hence

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}]=$$
$$\begin{bmatrix} \left(J_x c^2\varphi + J_y s^2\varphi\right) c^2\theta + J_z s^2\theta & \left(J_z - J_y\right) s\varphi c\varphi c\theta & J_z s\theta \\ \left(J_x - J_y\right) s\varphi c\varphi c\theta & J_x s^2\varphi + J_y c^2\varphi & 0 \\ J_z s\theta & 0 & \mathrm{J_z} \end{bmatrix}. \tag{40}$$

For the planar motion of the rigid solid, considering that the origin O of the mobile frame coincides with the center of weight, the mobile axes are central principal axes of inertia, and the motion takes place in the O_0XY plane, one may write

$$[\boldsymbol{\psi}]=[\mathbf{I}],\ [\boldsymbol{\theta}]=[\mathbf{I}],\ [\boldsymbol{\varphi}]=\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

;

$$[\mathbf{A}]=[\boldsymbol{\varphi}],\ [\mathbf{S}]=[\mathbf{0}]; \tag{42}$$

$$[\mathbf{J}_C]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J \end{bmatrix}; \tag{43}$$

$$[\mathbf{Q}]=\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{44}$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J \end{bmatrix}; \tag{45}$$

$$\{\boldsymbol{\omega}\}=\begin{bmatrix} 0 & 0 & \dot{\varphi} \end{bmatrix}^{\mathrm{T}},\ [\boldsymbol{\omega}]=\begin{bmatrix} 0 & -\dot{\varphi} & 0 \\ \dot{\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{46}$$

$$[\mathbf{M}]=\begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J \end{bmatrix}; \tag{47}$$

$$[\mathbf{r}_A]=\begin{bmatrix} 0 & 0 & y_A \\ 0 & 0 & -x_A \\ -y_A & x_A & 0 \end{bmatrix}; \tag{48}$$

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}=\begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}+[\mathbf{A}][\mathbf{r}_A]; \tag{49}$$

$$X_A = X_O + x_A\cos\varphi - y_A\sin\varphi,$$
$$Y_A = Y_O + x_A\sin\varphi + y_A\cos\varphi,\ Z_A = Z_O = 0; \tag{50}$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}}=$$
$$\begin{bmatrix} 0 & 0 & y_A\cos\varphi + x_A\sin\varphi \\ 0 & 0 & y_A\sin\varphi - x_A\cos\varphi \\ -y_A\cos\varphi - x_A\sin\varphi & -y_A\sin\varphi + x_A\cos\varphi & 0 \end{bmatrix}; \tag{51}$$

$$\{\mathbf{F_q}\}=\begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} F_{X_O} \\ F_{Y_O} \\ F_{Z_O} \end{bmatrix}=$$
$$\begin{bmatrix} F_{X_O} \\ F_{Y_O} \\ 0 \\ 0 \\ 0 \\ \left(-y_A\cos\varphi - x_A\sin\varphi\right)F_{X_O} + \left(-y_A\sin\varphi + x_A\cos\varphi\right)F_{Y_O} \end{bmatrix}; \tag{52}$$

$$\{\mathbf{s}\}=\begin{bmatrix} X_O & Y_O & 0 \end{bmatrix}^{\mathrm{T}},\ \{\boldsymbol{\beta}\}=\begin{bmatrix} 0 & 0 & \varphi \end{bmatrix}^{\mathrm{T}}; \tag{53}$$

$$\{\dot{\mathbf{s}}\}=\begin{bmatrix} \dot{X}_O & \dot{Y}_O & 0 \end{bmatrix}^{\mathrm{T}},\ \{\dot{\boldsymbol{\beta}}\}=\begin{bmatrix} 0 & 0 & \dot{\varphi} \end{bmatrix}^{\mathrm{T}}; \tag{54}$$

$$\{\widetilde{\mathbf{F}}_{\mathbf{s}}\}=\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}; \tag{55}$$

$$[\dot{\mathbf{Q}}]=\dot{\varphi}\begin{bmatrix}-\sin\varphi & \cos\varphi & 0\\ -\cos\varphi & -\sin\varphi & 0\\ 0 & 0 & 0\end{bmatrix}; \qquad (56)$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\dot{\mathbf{Q}}]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}; \qquad (57)$$

$$[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\omega}][\mathbf{J}_C][\mathbf{Q}]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}; \qquad (58)$$

$$\{\widetilde{\mathbf{F}}_{\boldsymbol{\beta}}\}=[0\quad 0\quad 0]^{\mathrm{T}}; \qquad (59)$$

$$\{\widetilde{\mathbf{F}}_{\mathbf{q}}\}=[0\quad 0\quad 0\quad 0\quad 0\quad 0]^{\mathrm{T}}; \qquad (60)$$

$$\{\mathbf{F}_{\mathbf{q}}\}+\{\widetilde{\mathbf{F}}_{\mathbf{q}}\}=\begin{bmatrix}F_{X_O}\\ F_{Y_O}\\ 0\\ 0\\ 0\\ (-y_A\cos\varphi-x_A\sin\varphi)F_{X_O}+\\ (-y_A\sin\varphi+x_A\cos\varphi)F_{Y_O}\end{bmatrix} \qquad (61)$$

and the matrix differential equation of motion takes the simplified form

$$\begin{bmatrix}\begin{bmatrix}m&0&0\\0&m&0\\0&0&J\end{bmatrix} & -[\mathbf{B}]^{\mathrm{T}}\\ [\mathbf{B}] & [\mathbf{0}]\end{bmatrix}\begin{bmatrix}\begin{bmatrix}\ddot{X}_O\\ \ddot{Y}_O\\ \ddot{\varphi}\end{bmatrix}\\ \{\boldsymbol{\lambda}\}\end{bmatrix}=$$

$$\begin{bmatrix}\begin{bmatrix}F_{X_O}\\ F_{Y_O}\\ (-y_A\cos\varphi-x_A\sin\varphi)F_{X_O}+\\ (-y_A\sin\varphi+x_A\cos\varphi)F_{Y_O}\end{bmatrix}\\ \{\dot{\mathbf{C}}\}-[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}\end{bmatrix}. \qquad (62)$$

Let us observe that if $[\mathbf{B}]=[\mathbf{0}]$, then one gets the equations of motion of the free rigid solid,

$$m\ddot{X}_O=F_{X_O},\ m\ddot{Y}_O=F_{Y_O},$$
$$J\ddot{\varphi}=$$
$$(-y_A\mathrm{c}\varphi-x_A\mathrm{s}\varphi)F_{X_O}+(-y_A\mathrm{s}\varphi+x_A\mathrm{c}\varphi)F_{Y_O}. \qquad (63)$$

6. EQUILIBRIUM POSITIONS

In this case one obtains the equation

$$\{\mathbf{F}_{\mathbf{q}}\}+[\mathbf{B}]^{\mathrm{T}}\{\boldsymbol{\lambda}\}=\{\mathbf{0}\}, \qquad (64)$$

at which one has to add the matrix function of constraints

$$\{\mathbf{f}(\{\mathbf{q}\})\}=\begin{bmatrix}f_1(q_1,q_2,\ldots,q_n)\\ \cdots\cdots\cdots\\ f_p(q_1,q_2,\ldots,q_n)\end{bmatrix}=\begin{bmatrix}0\\ \ldots\\ 0\end{bmatrix}. \qquad (65)$$

It results a system of non-linear equations from which one may determine the position of equilibrium and the reactions.

Denoting

$$\{\mathbf{F}_{X_O}\}=\frac{\partial\{\mathbf{F}_{\mathbf{q}}\}}{\partial X_O},\ \{\mathbf{F}_{Y_O}\}=\frac{\partial\{\mathbf{F}_{\mathbf{q}}\}}{\partial Y_O},\ \{\mathbf{F}_{Z_O}\}=\frac{\partial\{\mathbf{F}_{\mathbf{q}}\}}{\partial Z_O},$$
$$\{\mathbf{F}_{\psi}\}=\frac{\partial\{\mathbf{F}_{\mathbf{q}}\}}{\partial\psi},\ \{\mathbf{F}_{\theta}\}=\frac{\partial\{\mathbf{F}_{\mathbf{q}}\}}{\partial\theta},\ \{\mathbf{F}_{\varphi}\}=\frac{\partial\{\mathbf{F}_{\mathbf{q}}\}}{\partial\varphi}, \qquad (66)$$

$$[\mathbf{B}_{X_O}]=\frac{\partial[\mathbf{B}]^{\mathrm{T}}}{\partial X_O},\ [\mathbf{B}_{Y_O}]=\frac{\partial[\mathbf{B}]^{\mathrm{T}}}{\partial Y_O},$$
$$[\mathbf{B}_{Z_O}]=\frac{\partial[\mathbf{B}]^{\mathrm{T}}}{\partial Z_O},\ [\mathbf{B}_{\psi}]=\frac{\partial[\mathbf{B}]^{\mathrm{T}}}{\partial\psi},\ [\mathbf{B}_{\theta}]=\frac{\partial[\mathbf{B}]^{\mathrm{T}}}{\partial\theta}, \qquad (67)$$
$$[\mathbf{B}_{\varphi}]=\frac{\partial[\mathbf{B}]^{\mathrm{T}}}{\partial\varphi},$$

$$\{\mathbf{F}^*\}=[\{\mathbf{F}_{X_O}\}\ \ \ldots\ \ \{\mathbf{F}_{\varphi}\}], \qquad (68)$$

$$[\mathbf{B}^*]=[[\mathbf{B}_{X_O}]\{\boldsymbol{\lambda}\}\ \ \ldots\ \ [\mathbf{B}_{\varphi}]\{\boldsymbol{\lambda}\}] \qquad (69)$$

and using a variant in finite differences for the Newton-Raphson method, one gets the matrix equation

$$\begin{bmatrix}[\mathbf{F}^*]+[\mathbf{B}^*] & [\mathbf{B}]^{\mathrm{T}}\\ [\mathbf{B}] & [\mathbf{0}]\end{bmatrix}\begin{bmatrix}\{\Delta\mathbf{q}\}\\ \{\Delta\boldsymbol{\lambda}\}\end{bmatrix}=\begin{bmatrix}-\{\mathbf{F}_{\mathbf{q}}\}-[\mathbf{B}]^{\mathrm{T}}\{\boldsymbol{\lambda}\}\\ -\{\mathbf{f}(\{\mathbf{q}\})\}\end{bmatrix}. \qquad (70)$$

7. CONCLUSIONS

In our paper we presented a general multibody method by which one may study the dynamics of a rigid solid with general constraints and acted by conservative forces. The formulae obtained here can be directly used in practical problems. As one may easily see, the great challenge is to determine the column matrix $\{\mathbf{F_q}\}$. At this moment we made no assumption about the matrix of constraints

The paper is a generalization of the cases presented in [15] where the study is reduced to the equilibrium positions, these equilibrium positions being obtained from the second order Lagrange equations.

A great problem is the non-singularity of the left-hand term matrix in equation (32). If one wants to use this equation for the study of a planar case (instead of using equation (62)), then one may observe that the matrix is not an invertible one. A possible way to determine the motion is to eliminate some rows and columns from the left-hand term matrix in relation (32) to obtain an invertible matrix. A second variant is to directly use the relation (62). Even in the general case it is possible to obtain non-singular matrix (for instance, when the system has an useless degree of freedom).

Generally speaking, the mechanical system may have zero, one or multiple equilibrium positions. The convergence of the algorithm presented above for the determination of the equilibrium positions is assured only if the initial values are selected sufficiently closed to one of the equilibrium position. In this situation, different initial values may lead to different solutions or the algorithm may offer no solution (the algorithm is divergent).

In the second part of the paper we will present some examples in order to highlight the theory.

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