Modal Analysis of a Mechanical System Modeled as a 6 Degrees-of-Freedom Solid Body with Elastic Bearings and Structural Symmetries

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Abstract: - The paper presents the physical model and the mathematical model of a mechanical system modeled as a solid body with 6 degrees of dynamic freedom and elastic bearings. Considering that the system has constructive symmetries (masses and inertia, elastic support elements), simpler physical and mathematical models are obtained, the systems of differential equations of motion being decoupled in subsystems with fewer equations. The systems of decoupled equations of motion describe the dynamic behavior of subsystems with coupled motions in certain directions (translational and/or rotational motions). The paper also presents a case study: a modal analysis of an 6DOF elastic mechanical system with a vertical plane of symmetry.

Keywords: - modal analysis, elastic mechanical system, 6DOF, structural symmetry

1. INTRODUCTION. MATHEMATICAL MODEL OF 6DOF RIGID BODY WITH VISCOUS ELASTIC BEARINGS

The system of linear differential equations of motion of the solid body with visco-elastic bearings has six elastically and viscously coupled equations. In matrix form, the dynamic 6DOF model for the forced vibrations of this system is written [1-4]

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q^F\}\,, \qquad (1)$$

where:
$[M]$ is the 6×6 symmetrical matrix of inertia
$[C]$ - the 6×6 symmetrical matrix of damping
$[K]$ - the 6×6 symmetrical matrix of stiffness
$\{q\}$ - the 6-dim. column vector of displacements
$\{\dot{q}\}$ - the 6-dim. column vector of velocities
$\{\ddot{q}\}$ - the 6-dim. column vector of accelerations
$\{Q^F\}$ - the 6-dim. column vector of disturbing forces

2. EIGENPULSATIONS AND EIGENVECTORS

For the modal analysis of the elastic mechanical system, the system of differential equations of motion (1) is considered without damping, and undisturbed [5-8]:

$$\left.\begin{matrix}[C] \equiv [0] \\ \{Q^F\} \equiv \{0\}\end{matrix}\right\} \Rightarrow [M]\{\ddot{q}\} + [K]\{q\} = \{0\} \qquad (2)$$

If it multiplies to the left the eq. (3) with the inverse of the inertia matrix (which is non-singular), the canonical differential matrix equation is obtained as follows [9-11]

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\} \mid \times [M]^{-1}\ (left)$$

$$\Rightarrow \underbrace{[M]^{-1}[M]}_{I_6}\{\ddot{q}\} + \underbrace{[M]^{-1}[K]}_{[D]}\{q\} = \{0\}$$

$$\Rightarrow \{\ddot{q}\} + [D]\{q\} = \{0\} \qquad (3)$$

where:
$I_6 = DIAG(1,1,1,1,1,1)$ is the 6-dim identity matrix
$[D]_{6\times 6} = [M]^{-1}[K]$ - dynamic matrix of linear system

In order to determine the eigenmodes of vibration (eigenpulsations and eigenvectors) it considers the algebraic matrix equation [12-14]

$$([D]-\lambda I_6)\{v\}=\{0\}\ \rightarrow [D]\{v\}=\lambda\{v\}\ , \quad (4)$$

where:
$\lambda \equiv p^2$ is the eigenvalue of 6DOF system
p - the eigenpulsation of 6DOF system
$\{v\}$ - eigenvector/eigenshape of 6DOF system

The eq. (4) has six distinct solutions:

$$[D]\{v\}_i=\lambda_i\{v\}_i \ \ i=\overline{1,6}\ , \quad (5)$$

where:
$\left(p_i=\sqrt{\lambda_i}\,,\{v\}_i\right)\ i=\overline{1,6}$ are the six modes of the system

3. INERTIA AND STIFFNESS MATRICES OF THE LINEAR SYSTEM

Considering a reference system **Oxyz** with the origin in the center of gravity of the rigid body with elastic bearings **O≡C** and central and principal axes **Cx**, **Cy**, **Cz** (centrifugal moments of inertia are zero), the matrix of inertia becomes diagonal [2] [4] [14] [15]

$$[M]=DIAG\left(m,m,m,J_{xx},J_{yy},J_{zz}\right)\ , \quad (6)$$

where
- m is the mass of the solid body
- J_{xx}, J_{yy} and J_{zz} are the central and principal moments of inertia.

Neglecting dampings, the mathematical model of the system is elastic coupled through the matrix of the stiffness coefficients $[K]$; differential equations of motion remain elastically coupled. If the solid body has ***n*** elastic bearings with triorthogonal stiffness $\left(k_{jx},k_{jy},k_{jz}\right)\ j=\overline{1,n}$, in the points of coordinates $N\left(x_j,y_j,z_j\right)\ j=\overline{1,n}$, the elastic coupling coefficients matrix is

$$[K]=\begin{bmatrix}[K]_{11} & [K]_{12}\\ [K]_{21} & [K]_{22}\end{bmatrix}\ , \quad (6)$$

where the stiffness submatrices are [3] [16] [17]

$$[K]_{11}=\begin{bmatrix}\sum k_{jx} & 0 & 0\\ 0 & \sum k_{jz} & 0\\ 0 & 0 & \sum k_{jz}\end{bmatrix} \quad (7)$$

$$[K]_{12}=\begin{bmatrix}0 & \sum k_{jx}\,z_j & -\sum k_{jx}\,y_j\\ -\sum k_{jy}\,z_j & 0 & \sum k_{jy}\,x_j\\ \sum k_{jz}\,y_j & -\sum k_{jx}\,x_j & 0\end{bmatrix} \quad (8)$$

$$[K]_{21}=\begin{bmatrix}0 & -\sum k_{jy}\,z_j & \sum k_{jz}\,y_j\\ \sum k_{jx}\,z_j & 0 & -\sum k_{jx}\,x_j\\ -\sum k_{jx}\,y_j & \sum k_{jy}\,x_j & 0\end{bmatrix} \quad (9)$$

$$[K]_{22}=\begin{bmatrix}\sum\left(k_{jy}z_j^2+k_{jz}y_j^2\right) & -\sum k_{jz}\,x_jy_j & -\sum k_{jy}\,z_jx_j\\ -\sum k_{jz}\,x_jy_j & \sum\left(k_{jz}x_j^2+k_{jx}z_j^2\right) & -\sum k_{jx}\,y_jz_j\\ -\sum k_{jy}\,z_jx_j & -\sum k_{jx}\,y_jz_j & \sum\left(k_{jx}y_j^2+k_{jy}x_j^2\right)\end{bmatrix} \quad (10)$$

4. THE MOVEMENTS OF THE SYSTEM WITH STRUCTURAL SYMMETRIES

Figure 1 shows the model of the elastic mechanical system with structural symmetries (inertia, elasticity, damping, bearings positions) and central and principal axis system **Cxyz**. The mechanical elastic system has a vertical-longitudinal plane of symmetry **yCz** if [19] [22] [23]:

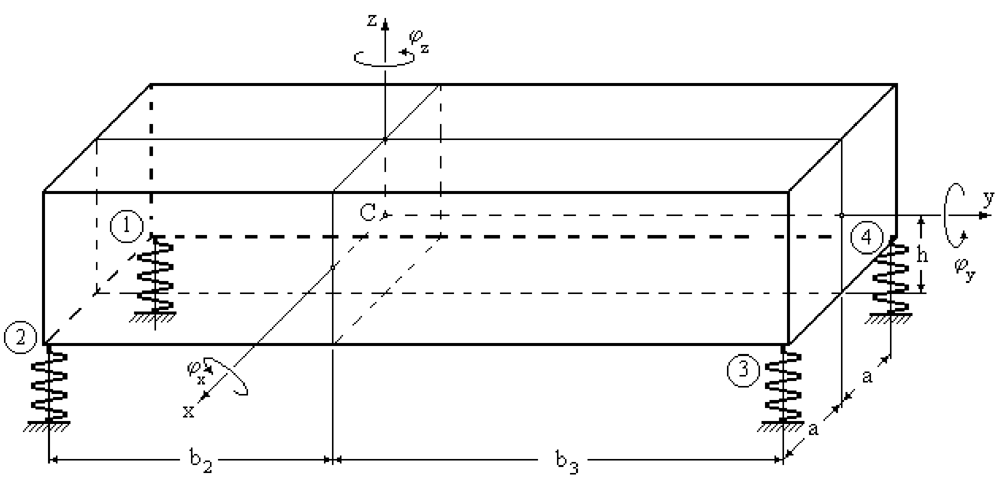

Figure 1. Rigid body 6DOF with structural vertical-longitudinal plane of symmetry [3] [10] [14]

I)the pairs of support elements 1-2 and 3-4 are identical but 2-3 and 4-1 are not OR
II)$b_2 \neq b_3$ OR
III)both of them.

If the system has structural symmetry on the vertical-longitudinal plane **yCz**, the six movements are decoupled into two subsystems with three coupled movements as follows [20] [21]:

1) the subsystem 1 with coupled movements of:
   1.1) lateral slip/skidding - coordinate X
   1.2) rolling rotation- coordinate φ_Y
   1.3) gyration/turning rotation- coordinate φ_Z
2) the subsystem 2 with coupled movements of:
   2.1) forward movement - coordinate Y
   2.2) vertical/jumping movement- coordinate Z
   2.3) pitching rotation- coordinate φ_X

The differential moving eq. of free vibrations for each subsystems are: [3] [7] [18] [20]

1) for the subsystem 1 $(X, \varphi_Y, \varphi_Z)$

$$\begin{cases} m\ddot{X} + 4k_xX - 4hk_x\varphi_Y - 2k_x(b_3 - b_2)\varphi_Z = 0 \\ J_{yy}\ddot{\varphi}_Y - 4hk_xX + 4(h^2k_x + a^2k_z)\varphi_Y + \\ \qquad +2k_x(b_3 - b_2)\varphi_Z = 0 \\ J_{zz}\ddot{\varphi}_Z - 2k_x(b_3 - b_2)X + 2k_x(b_3 - b_2)\varphi_Y + \\ \qquad +2[2a^2k_y + (b_3^2 + b_2^2)k_x]\varphi_Z = 0 \end{cases} \quad (11)$$

2) for the subsystem 1 $(X, \varphi_Y, \varphi_Z)$

$$\begin{cases} m\ddot{Y} + 4k_yY + 4hk_y\varphi_X = 0 \\ m\ddot{Z} + 4k_zZ + 2k_z(b_3 - b_2)\varphi_X = 0 \\ J_{xx}\ddot{\varphi}_X + 4hk_yY + 2k_z(b_3 - b_2)Z + \\ \qquad +2[2h^2k_y + (b_3^2 + b_2^2)k_z]\varphi_X = 0 \end{cases} \quad (12)$$

5. CASE STUDY - MODAL ANALYSIS OF A CONCRETE BEAM BEARED ON FOUR ELASTIC SUPPORTS

Figure 2 shows a reinforced concrete beam used for the deck structure of some bridges on the Romanian highways. The beam is beared in four points (**1**, **2**, **3** and **4**) on elastic support devices, identical two by two at each of the 2 ends (**1** with **2**, **3** with **4**), so that the beam-bearing system has a vertical-longitudinal plane of symmetry. The dimensional and structural characteristics of the system are as follows (from execution and assembly drawings or calculated):

1) inertia characteristics

-mass $m = 83 \times 10^3\ kg$
-position of gravity center **C** reported to horizontal plane of bearing points (calculated) $h = 1031\ mm$
-principal axial moments of inertia (calculated):

$$J_{xx} = 11.105 \times 10^6\ kgm^2$$
$$J_{yy} = 0.0491 \times 10^6\ kgm^2$$
$$J_{zz} = 11.064 \times 10^6\ kgm^2$$

2) dimensional characteristics

-sizes L×W×H $40000 \times 1200 \times 2000\ mm$
-supports mounting sizes

$$2a = 390\ mm$$
$$B(= b_2 + b_3) = 39100\ mm$$
$$b_2 = b_3 = 19550\ mm$$

3) directional stiffness of the elastic supports

Table 1. Stiffness of elastic supports

Support	Stiffness *k* [×10^6 N/m]		
	k_x	k_y	k_z
1	0.26	0.26	312
2	0.26	0.26	312
3	0.51	0.51	496
4	0.51	0.51	496

4) the coordinates of the bearings (in the coordinates system **Cxyz***)*

Table 2. Coordinates of the bearings

Bearing	Coordinate [m]		
	x	*y*	*z*
1	-0.195	-19.550	-1.031
2	0.195	-19.550	-1.031
3	0.195	19.550	-1.031
4	-0.195	19.550	-1.031

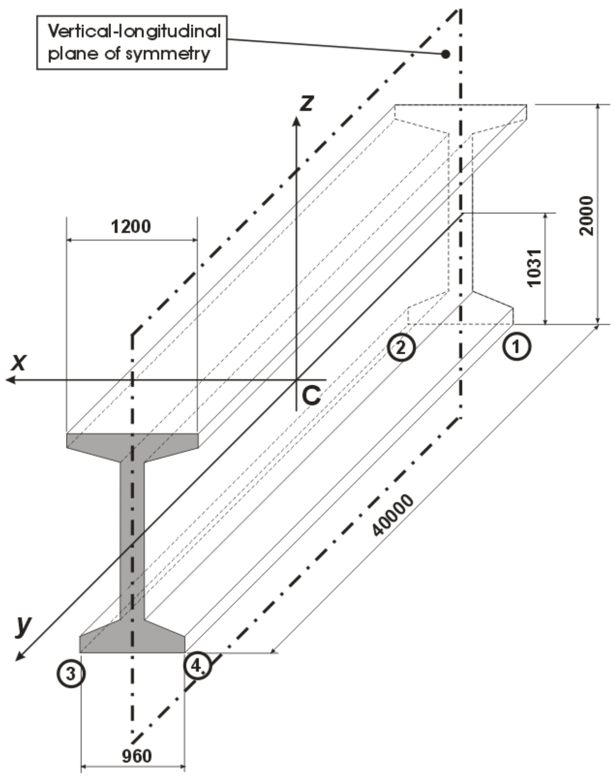

Figure 2. 6DOF model of beam with elastic supports

5.1. Inertia matrices, stiffness matrices and dynamic matrices of decoupled subsystems

5.1.1. System $(\boldsymbol{X}, \boldsymbol{\varphi_Y}, \boldsymbol{\varphi_Z})$

$$[M_1] = 10^3 \begin{bmatrix} 83 & 0 & 0 \\ 0 & 49.1 & 0 \\ 0 & 0 & 11064 \end{bmatrix} \begin{bmatrix} kg & kgm & kgm \\ kgm & kgm^2 & kgm^2 \\ kgm & kgm^2 & kgm^2 \end{bmatrix}$$

$$[K_1] = 10^3 \begin{bmatrix} 1540 & -1588 & -9775 \\ -1588 & 63085 & 10077 \\ -9775 & 10077 & 588650 \end{bmatrix} \begin{bmatrix} N/m & N & N \\ N & Nm & Nm \\ N & Nm & Nm \end{bmatrix}$$

$$[D_1] = [M_1]^{-1}[K_1] = \begin{bmatrix} 18.565 & -19.140 & -117.842 \\ -32.332 & 1284.66 & 205.222 \\ -0.883 & 0.9108 & 53.203 \end{bmatrix} s^{-2}$$

5.1.2. System $(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{\varphi_x})$

$$[M_2] = 10^3 \begin{bmatrix} 83 & 0 & 0 \\ 0 & 83 & 0 \\ 0 & 0 & 11105 \end{bmatrix} \begin{bmatrix} kg & kg & kgm \\ kg & kg & kgm \\ kgm & kgm & kgm^2 \end{bmatrix}$$

$$[K_2] = 10^3 \begin{bmatrix} 1540 & 0 & 1588 \\ 0 & 1616000 & 7194400 \\ 1588 & 7194400 & 61764000 \end{bmatrix} \begin{bmatrix} N/m & N & N \\ N & Nm & Nm \\ N & Nm & Nm \end{bmatrix}$$

$$[D_2] = [M_2]^{-1}[K_2] = \begin{bmatrix} 18.565 & 0 & 19.1403 \\ 0 & 19481.5 & 86731.3 \\ 0.14297 & 647.854 & 55618.5 \end{bmatrix} s^{-2}$$

5.2. Eigenvalues, eigenpulsations and eigenvectors of decoupled subsystems

Eigenvalues were determined by solving the characteristic polynomial equation using numerical calculation software Matlab®:

$$det([D] - \lambda[I_n]) = 0 \qquad (13)$$

Eigenshapes of the two decoupled subsystems were determined by solving the matrix eq. (5) for each of them.

5.2.1. System $(\boldsymbol{X}, \boldsymbol{\varphi_Y}, \boldsymbol{\varphi_Z})$

$$\begin{bmatrix} 18.565 - \lambda & -19.140 & -117.842 \\ -32.332 & 1284.66 - \lambda & 205.222 \\ -0.883 & 0.9108 & 53.203 - \lambda \end{bmatrix} = 0\ [s^{-2}]$$

Table 3. Eigenshapes of system $(X, \varphi_Y, \varphi_Z)$

Mode parameter	Eigenshape		
	Mode 1	***Mode 2***	***Mode 3***
$\lambda\ [s^{-2}]$	15.4526	5577.09	1285.30
$p\ [rad/s]$	3.93	7.46	35.85
$f = p/2\pi\ [Hz]$	0.62	1.12	5.71
$\{v_i\}$	$\begin{Bmatrix} 1 \\ 0.0218 \\ 0.0229 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0.8109 \\ -0.3280 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ -65.878 \\ -0.0494 \end{Bmatrix}$

$$\begin{bmatrix} 18.565 & 0 & 19.1403 \\ 0 & 19481.5 & 86731.3 \\ 0.14297 & 647.854 & 55618.5 \end{bmatrix}$$

5.2.2. System $(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{\varphi_x})$

$$\begin{bmatrix} 18.565 - \lambda & 0 & 19.1403 \\ 0 & 19481.5 - \lambda & 86731.3 \\ 0.14297 & 647.854 & 55618.5 - \lambda \end{bmatrix} = 0\ [s^{-2}]$$

Table 4. Eigenshapes of system (Y, Z, φ_x)

Mode parameter	Eigenshape		
	Mode 4	***Mode 5***	***Mode 6***
$\lambda\ [s^{-2}]$	18.5652	5577.09	1285.30
$p\ [rad/s]$	4.31	7.46	35.85
$f = p/2\pi\ [Hz]$	0.69	1.12	5.71
$\{v_i\}$	$\begin{Bmatrix} 1 \\ 0.000012 \\ -0.000003 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ -54539.2 \\ 938.967 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 6880.11 \\ 2985.07 \end{Bmatrix}$

6. CONCLUSIONS

1) modeling a solid body with elastic bearings and various types of symmetries leads to decoupled differential equations of motion with fewer coupling coefficients and, therefore, easier to study analytically; in this way, the influences of the dimensional, inertial and elastic parameters on the forms of the modes of vibration can be highlighted;

2) if the movements of a solid body with symmetries are relate to a system of central and principal axes **Cxyz**, then its movements along the six “directions” $(X, Y, Z, \varphi_X, \varphi_Y, \varphi_Z)$ are coupled only by the non-diagonal coefficients of the stiffness matrix;

3) for the case study considered, the values either very high or very low of the coefficients of the eigenvectors lead to the conclusion that, inside the subsystems with coupled motions, the couplings are "weak"; in fact, the movements of these subsystems can also be considered to be quasi-decoupled.

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