# On the Reduction of a System of Planar Cylindrically Jointed Bars with Small Deformations

**Maria-Luiza BEȘLIU-GHERGHESCU**
*University of Pitești, street Târgul din Vale, no 1, 110040, ROMANIA, e-mail: besliuluiza@yahoo.com*

**Nicolae-Doru STĂNESCU**
*University of Pitești, street Târgul din Vale, no 1, 110040, ROMANIA, e-mail: s_doru@yahoo.com*
*Member SRA*

*Abstract:* - In this paper we present a general method to obtain the equivalent system for an arbitrary planar system of cylindrically jointed bar. The equivalent system is one for which the displacement of the origin (the point at which all bars are jointed) is equal to the displacement of the same point for the original system. It is proved that, in general, any system of bars fulfilling some additional conditions is equivalent to a system of two bars. This equivalence is valid only for small deformations of the system. An example highlights the theory.

*Keywords:* - small displacements, equivalent system

## 1. INTRODUCTION

A general system of planar bars may be considered from two approaches. The first approach assumes that the deformations of bars are small, so some linear approximations may be performed [1-6, 8]. The second approach considers that the deformations of bars may be arbitrary, the only condition being that the linear relation between the forces and the deformations holds true [7, 10]. In the last situation some additional assumptions referring to the shape of the deformed bars (when not all of them are cylindrically jointed) are made in order to simplify the system of equations. Even if all the bars are cylindrically jointed the obtained system is no longer a linear one and there exist many difficulties for its solving.

In this paper we discuss the problem of the equivalent system of an arbitrary system of planar bars cylindrically jointed. We define the equivalent system that one with a minimum number of bars, which present the same displacements of the point at which the bars are jointed as the original system. It is obvious that this equivalent system must contain at least two bars. We will prove that in the general case two bars are sufficient in order to characterize the equivalent system

In a previous paper [11] we have studied the vibrations of such arbitrary planar system of bars. The main assumption is that there exists no pair of bars situated on the same straight line; this assumption is also considered in this paper.

## 2. MECHANICAL SYSTEM

One considers the system in Fig.1 consisting in $n$ bars $OA_i$, $i=\overline{1,n}$, for which one knows: the length of each bar $l_i$, the elasticity modulus of each bar $E_i$, and the cross-sectional area of each bar $A_i$. In addition, the angles $\alpha_i$ formed by the directions $OA_i$ of the bar $i$ and the positive direction of the $Ox$ - axis are given and $\alpha_i \neq \alpha_j$, $\alpha_i \neq \alpha_j + \pi$ for any $i, j=\overline{1,n}$, $i \neq j$.

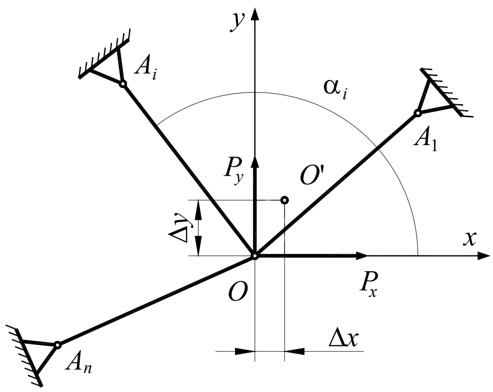


**Figure 1.** The mechanical system

The bars are cylindrically jointed at the points $A_i$ and at the point $O$.

The system is acted by a force $\mathbf{P}$ of components $P_x$ and $P_y$ at the point $O$. In these conditions, the system is deformed, the point $O$ arriving at the point $O'$, the displacements being $\Delta x$ and $\Delta y$.

## 3. THE EQUIVALENT SYSTEM

The displacements $\Delta x$ and $\Delta y$ are obtained from the linear system

$$\left(\sum_{i=1}^{n} k_i \cos^2 \alpha_i\right)\Delta x + \left(\sum_{i=1}^{n} k_i \cos\alpha_i \sin\alpha_i\right)\Delta y = P_X \, , \quad (1)$$
$$\left(\sum_{i=1}^{n} k_i \cos\alpha_i \sin\alpha_i\right)\Delta x + \left(\sum_{i=1}^{n} k_i \sin^2 \alpha_i\right)\Delta y = P_y \, ,$$

where

$$k_i = \frac{E_i A_i}{l_i} \, . \quad (2)$$

We will denote

$$A = \sum_{i=1}^{n} k_i \cos^2 \alpha_i \, , \; B = \sum_{i=1}^{n} k_i \cos\alpha_i \sin\alpha_i \, , \quad C = \sum_{i=1}^{n} k_i \sin^2 \alpha_i \quad (3)$$

and the system (1) becomes

$$A\Delta x + B\Delta y = P_x \, , \; B\Delta x + C\Delta y = P_y \, , \quad (4)$$

that is, a linear system of two equations with two unknowns.

The determinant $\Delta$ of the system (4) reads

$$\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 \quad (5)$$

First of all, we will prove that $\Delta > 0$.

Taking into account the notations, it results

$$\Delta = \left(\sum_{i=1}^{n} k_i \cos^2 \alpha_i\right)\left(\sum_{i=1}^{n} k_i \sin^2 \alpha_i\right) - \left(\sum_{i=1}^{n} k_i \cos\alpha_i \sin\alpha_i\right)^2 = \sum_{i=1}^{n}\sum_{j=i+1}^{n} k_i k_j \sin^2\left(\alpha_i - \alpha_j\right) \quad (6)$$

and since $\alpha_i \neq \alpha_j$, $\alpha_i \neq \alpha_j + \pi$, for any $i \neq j$, one gets $\Delta > 0$.

We will prove now that the system reduces to a system with only two bars.Let us denote these bars by 1 and 2, their elastic parameters being $k_1$ and $k_2$, and the bars form the angles $\theta_1$ and $\theta_2$ with the positive direction of the $Ox$ - axis.

The system of two bars leads to the displacements

$$\begin{aligned} &\left(k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2\right)\Delta x' + \\ &+ \left(k_1 \cos\theta_1 \sin\theta_1 + k_2 \cos\theta_2 \sin\theta_2\right)\Delta y' = P_x \, , \\ &\left(k_1 \cos\theta_1 \sin\theta_1 + k_2 \cos\theta_2 \sin\theta_2\right)\Delta x' + \\ &+ \left(k_1 \sin^2 \theta_1 + k_2 \sin^2 \theta_2\right)\Delta y' = P_y \, . \end{aligned} \quad (7)$$

The systems (4) and (7) have the same solution if and only if

$$k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2 = A \, , \quad (8)$$
$$k_1 \cos\theta_1 \sin\theta_1 + k_2 \cos\theta_2 \sin\theta_2 = B \, , \quad (9)$$
$$k_1 \sin^2 \theta_1 + k_2 \sin^2 \theta_2 = C \, . \quad (10)$$

Summing the equations (8) and (10) one gets

$$k_1 + k_2 = A + C \, , \quad (11)$$

while calculating the determinant, it results

$$k_1 k_2 \sin^2\left(\theta_2 - \theta_1\right) = AC - B^2 \, . \quad (12)$$

Further on, we will assume that $\sin^2\left(\theta_2 - \theta_1\right) = 1$, that is $\theta_2 - \theta_1 = \frac{\pi}{2}$, or $\theta_2 - \theta_1 = \frac{3\pi}{2}$. It results

$$k_1 + k_2 = A + C \, , \; k_1 k_2 = AC - B^2 \, , \quad (13)$$

wherefrom

$$k_1 = \frac{A + C + \sqrt{(A - C)^2 + 4B^2}}{2} \, , \quad k_2 = \frac{A + C - \sqrt{(A - C)^2 + 4B^2}}{2} \quad (14)$$

Let us observe that $k_1 > 0$ and $k_2 > 0$.

Further on, we will consider that $\theta_2 = \theta_1 + \frac{\pi}{2}$.

One gets

$$\cos\theta_2 = -\sin\theta_1 \, , \; \sin\theta_2 = \cos\theta_1 \quad (15)$$
$$k_1 \cos^2 \theta_1 + k_2 \cos^2 \theta_2 = k_1 \cos^2 \theta_1 + k_2 \sin^2 \theta_1 \quad (16)$$
$$k_1 \sin^2 \theta_1 + k_2 \sin^2 \theta_2 = k_1 \sin^2 \theta_1 + k_2 \cos^2 \theta_1 \, , \quad (17)$$
$$k_1 \cos\theta_1 \sin\theta_1 + k_2 \cos\theta_2 \sin\theta_2 = k_1 \cos\theta_1 \sin\theta_1 - k_2 \cos\theta_1 \sin\theta_1 \, . \quad (18)$$

From the condition

$$k_1 \cos^2 \theta_1 + k_2 \sin^2 \theta_1 = A \quad (19)$$

it results

$$\left(k_1 - k_2\right)\cos^2 \theta_1 = A - k_2 \, , \quad (20)$$

wherefrom

$$\cos\theta_1 = \pm\sqrt{\frac{A - k_2}{k_1 - k_2}} \, . \quad (21)$$

From the condition

$$\left(k_1 - k_2\right)\cos\theta_1 \sin\theta_1 = \frac{k_1 - k_2}{2}\sin\left(2\theta_1\right) = B \quad (22)$$

one gets

$$\sin 2\theta_1 = \frac{2B}{k_1 - k_2} \quad (23)$$

We have to prove the relations

$$0 \le \frac{A-k_2}{k_1-k_2} \le 1 \tag{24}$$

and

$$-1 \le \frac{2B}{k_1-k_2} \le 1. \tag{25}$$

Expression (24) is equivalent to

$$0 \le \frac{A-C+\sqrt{(A-C)^2+4B^2}}{2\sqrt{(A-C)^2+4B^2}} \le 1, \tag{26}$$

wherefrom

$$A-C+\sqrt{(A-C)^2+4B^2} \ge 0, \quad A-C \le \sqrt{(A-C)^2+4B^2}, \tag{27}$$

which, obviously, hold true.

The condition (25) leads to

$$-\sqrt{(A-C)^2+4B^2} \le 2B \le \sqrt{(A-C)^2+4B^2}, \tag{28}$$

which are obvious.

## 4. PARTICULAR CASES

i) The first particular case is defined by $B=0$. From the relations (14) one obtains either $k_1=0$, or $k_2=0$. Let us assume that $k_2=0$. The expressions (8), (9) and (10) offer

$$k_1\cos^2\theta_1 = A,\ k_1\cos\theta_1\sin\theta_1 = 0,\ k_1\sin^2\theta_1 = C, \tag{29}$$

wherefrom $\theta_1=0$, $\theta_1=\frac{\pi}{2}$, $\theta_1=\pi$, or $\theta_1=\frac{3\pi}{2}$ and, consequently $A=0$ or $C=0$; it results $\Delta=0$ and from expression (6) one gets $\alpha_i=\alpha_1$ or $\alpha_i=\alpha_1+\pi$, for all $i\ge 2$, which is impossible, according to our main assumption that the exists no pair of bars situated on the same straight line.

ii) The second case is characterized by $A=C$. From the expressions (14) one gets

$$k_1 = A+|B|,\ k_2 = A-|B|, \tag{30}$$

$$A-k_2=|B|,\ k_1-k_2=2|B|, \tag{31}$$

$$\cos\theta_1 = \pm\frac{\sqrt{2}}{2},\ \sin 2\theta_1 = \frac{2B}{2|B|} = \pm 1, \tag{32}$$

that is $\theta_1=\frac{\pi}{4}$ or $\theta_1=\frac{3\pi}{4}$.

## 5. EXAMPLE

For the system in Fig. 2 one knows $k_1=k$, $k_2=2k$, $k_3=\frac{k}{2}$, $k_4=\frac{3k}{2}$, and the angles $\alpha_1=\frac{\pi}{4}$, $\alpha_2=\frac{2\pi}{3}$, $\alpha_3=\frac{7\pi}{6}$, $\alpha_4=\frac{11\pi}{6}$, wherefrom

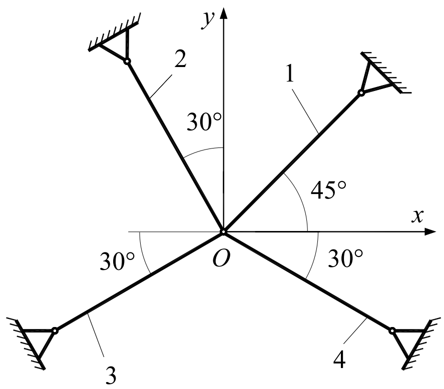


**Figure 2.** Example.

$$\cos\alpha_1=\frac{\sqrt{2}}{2},\ \sin\alpha_1=\frac{\sqrt{2}}{2},\ \cos^2\alpha_1=\frac{1}{2},\ \sin^2\alpha_1=\frac{1}{2},\ \cos\alpha_1\sin\alpha_1=\frac{1}{2}, \tag{33}$$

$$\cos\alpha_2=-\frac{1}{2},\ \sin\alpha_2=\frac{\sqrt{3}}{2},\ \cos^2\alpha_2=\frac{1}{4},\ \sin^2\alpha_2=\frac{3}{4},\ \cos\alpha_2\sin\alpha_2=-\frac{\sqrt{3}}{4}, \tag{34}$$

$$\cos\alpha_3=-\frac{\sqrt{3}}{2},\ \sin\alpha_3=-\frac{1}{2},\ \cos^2\alpha_3=\frac{3}{4},\ \sin^2\alpha_3=\frac{1}{4},\ \cos\alpha_3\sin\alpha_3=\frac{\sqrt{3}}{4}, \tag{35}$$

$$\cos\alpha_4=\frac{\sqrt{3}}{2},\ \sin\alpha_4=-\frac{1}{2},\ \cos^2\alpha_4=\frac{3}{4},\ \sin^2\alpha_4=\frac{1}{4},\ \cos\alpha_4\sin\alpha_4=-\frac{\sqrt{3}}{4}. \tag{36}$$

It successively results

$$A=\sum_{i=1}^{4}k_1\cos^2\alpha_i=\frac{1}{2}k+\frac{1}{4}\cdot 2k+\frac{k}{2}\cdot\frac{3}{4}+\frac{3k}{2}\cdot\frac{3}{4}=\frac{5k}{2}=2.5k, \tag{37}$$

$$B=\sum_{i=4}^{n} k_1 \cos\alpha_i \sin\alpha_i = \frac{1}{2}k-\frac{\sqrt{3}}{4}\cdot 2k$$
$$+\frac{\sqrt{3}}{4}\cdot\frac{k}{2}-\frac{\sqrt{3}}{4}\cdot\frac{3k}{2}=k\left(\frac{1}{2}-\frac{3\sqrt{3}}{4}\right) \tag{38}$$
$$=-0.799038k,$$
$$C=A. \tag{39}$$

Since $A=C$ one gets

$$k_1=(2.5+0.799038)k=3.299038k, \tag{40}$$
$$k_2=(2.5-0.799038)k=1.700962k, \tag{41}$$
$$\theta_1=\frac{\pi}{4},\ \theta_2=\frac{3\pi}{4} \tag{42}$$

or

$$\theta_1=\frac{3\pi}{4},\ \theta_2=\frac{5\pi}{4}. \tag{43}$$

The original system is equivalent to one of the system captured in Fig. 3, a) or 3, b).

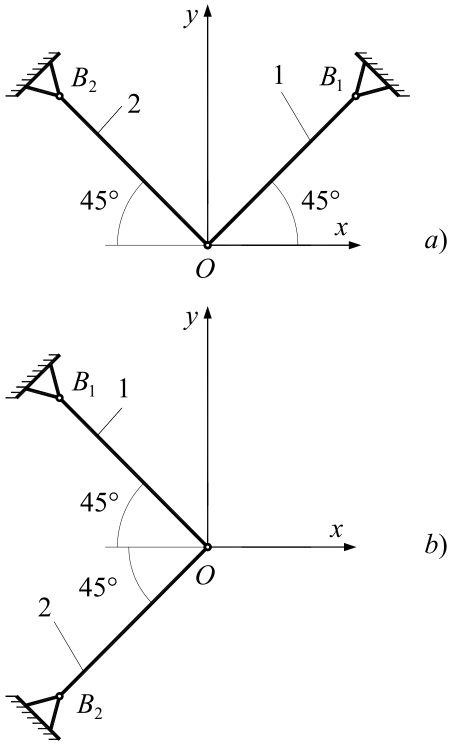


**Figure 3.** Solution of the Example.

## 6. DISCUSSIONS AND CONCLUSIONS

The calculations are performed for small deformations of the bars both for the initial and the equivalent systems of bars. The equivalent system is not unique. One may observe that we assumed $\sin^2(\theta_2-\theta_1)=1$, which is not a necessary condition. Depending on the value chosen for $\sin^2(\theta_2-\theta_1)$ one may obtain another equivalent system. The case $\sin(\theta_2-\theta_1)=0$ is not considered here because it imposes supplementary conditions for the initial system. The condition $B\neq 0$ is not imposed from the beginning. It was a consequence of the mathematical calculation.

The solutions presented in Fig. 3 are not the only solutions for $\sin^2(\theta_2-\theta_1)=1$. The other two solutions are the symmetrical of the first two with respect to the $Ox$ - axis.

It is also possible to impose the angles $\theta_1$ and $\theta_2$ resulting the values $k_1$ and $k_2$, where $\theta_2\neq\theta_1$, $\theta_2\neq\theta_1+\pi$.

Usually the positions of the new bars are given, that is one knows the jointing points $B_1$ and $B_2$, and the lengths $l_1$ and $l_2$ of these new two bars. One calculates the elastic parameters $k_1$ and $k_2$, and obtains the values $E_1A_1$ and $E_2A_2$, ($E_iA_i=k_il_i$, $i=\overline{1,2}$). Since the values $E_i$ are known, it results the cross-sectional areas of the bars.

The calculations are similar for bars composed from different materials.

One big challenge is the following one: does the equivalent system satisfy the conditions of small deformations and the tensions in the two bars do not exceed the admissible tensions? This is an open question which will be discussed in our future works.

## REFERENCES


[1] Bedford A., Liechti K.M., *Mechanics of Materials*, 2nd ed., Springer, 2019.

[2] Slocum S.E., Hancock E.L., *Text-book on the strength of materials*, Pala Press, 2015.

[3] Alemayehu K., *Strength of Materials*, Lambert, 2019.

[4] Timoshenko S., *Strength of Materials*, part 1 and 2, 3rd ed., Krieger Pub. Co., 1983.

[5] Hibbeler R.C., *Mechanics of Materials*, 8th ed., Prentice Hall, 2010.

[6] Gunneswara T.D., Andal M., *Strength of Materials: Fundamentals and Applications*, Cambridge, 2018.

[7] Fertis, D. G., *Nonlinear Mechanics*. CRC Press, Boca Raton, 1999.

[8] Jiao P., Alavi A. H., Borchani W., Lajnef N., Small and large deformation models of post-buckled beams under lateral constraints, *Mathematics and Mechanics of Solids*, Vol. 24, No. 2, 2019, pp. 386-405 (2019).

[9] Huang X., Yu T. X., Lu G., Lippmann, H., Large deflection of elastoplastic beams with prescribed moving and rotating ends, *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, Vol. 217, 2003, pp. 1001-1013.

[10] Kang Y. A., Li X. F., Large Deflections of a Non-linear Cantilever Functionally Graded Beam, *Journal of Reinforced Plastics and Composites*, Vol. 29, No. 12, 2010, pp. 1761-1774.

[11] Beşliu-Gherghescu, M. L., Pandrea, N., Stănescu, N.-D., On the Vibrations of a Planar System of Bars Cylindrically Jointed, RJAV, Vol. 16, No. 2, 2019, pp. 189-197