Optimization of Musical Non-Symmetric 3-Strings

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Abstract: - We develop a mathematical method to determine the physical features of non-symmetric 3-strings maximizing the degree of periodicity of their vibrations, which are generally inharmonic. The method is exemplify with two classes of 3-strings whose optimal configurations could be used to build new musical instruments.

Keywords: - optimization, differential equations on string networks, musical instruments

1. INTRODUCTION

The use of mathematics to describe and understand musical sounds begun with the Pythagoreans more than 2500 years ago. The modern mathematical analysis of sounds produced by stringed instruments, such as violins, or pianos, is largely based on the wave equation introduced by d'Alembert in 1746 [1]. These sounds are generated by small amplitude transverse oscillations of a certain number of strings fixed at each of their two extremities. The mathematical predictions deduced from this equation are in good agreement with observations (see e.g. [2]).

In this paper, we consider the idea of building musical instruments with networks of strings rather than two-extremity strings. Our objective is to optimize such a vibrating system, which is not to optimize the shape of the musical instrument itself (see e.g. [3]).

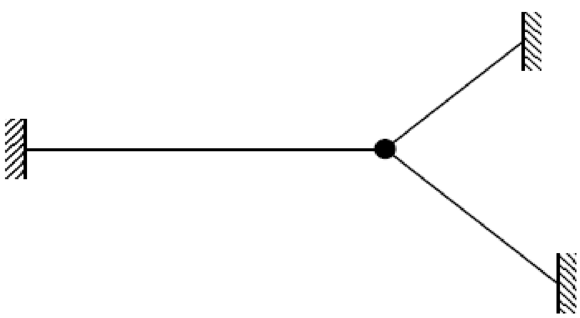

Figure 1. Schematic of a non-symmetric 3-string. Its three extremities connected to a section of wall are fixed, whereas the junction point of its three strings, here represented by a black dot, is free to move transversely to the plane network.

The simplest network of strings is a 3-string. We call 3-string any plane network of three stretched strings tied together at one common extremity [4-5]. This junction point is free to move transversely to the plane network while all its other extremities are fixed (Figure 1). It is possible to build a 3-string with sections of string having different lengths and different mass densities. We assume that each of these sections of string is subject to no internal nor external friction, and is perfectly flexible and with elasticity satisfying Hooke's law. As for almost all musical instruments built with ordinary strings, one can study the sounds produced by small-amplitude transverse vibrations of a 3-string through Fourier analysis.

We say that a 3-string is symmetric if it is built with three identical sections of strings with the same tension. Using mathematics, it is possible to show that the vibrations of such a 3-string can be as harmonic as any ordinary string [5]. However, a perfectly symmetric 3-string is physically impossible to realize, as it requires first to manufacture three identical sections of string, and then to join them so as they form three angles equal to $2\pi/3$. Moreover, even if this was possible, or if we were satisfied with an approximation of this perfect symmetry, the geometric configuration of such symmetric networks would make it very difficult to use them to build an ergonomic musical instrument.

Musical instruments using non-symmetric 3-strings produce sounds with inharmonic frequencies as a rule, which is generally unappealing to the human ear. The present paper aims to present and exemplify a mathematical method to determine the values of physical parameters allowing the design of musical instruments built with non-symmetric 3-strings that should produce sounds with maximal harmonicity, i.e. vibrations with a maximal degree of periodicity. From the musical point of view, this means that the sounds produced by an instrument made with such 3-strings would be closer to those made with ordinary strings, while keeping in the hands of their users the possibility to produce inharmonic sound effects, which appeal to a large class of music listeners [6].

We have organized the rest of this paper as follows. In Section 2, we set up the mathematical problem related to the description of small-amplitude transverse vibrations of any 3-string. We shall determine the configurations of maximal harmonicity among non-symmetric 3-strings by allowing the variation of four sets of factors acting upon their modes: (i) the mass density of each of the three strings; (ii) their lengths; (iii) the three angles between these strings, which influence their tensions; (iv) the value of a mass added to the junction point of the three strings [7]. Taking into account this last factor results from the observation that adding such a mass to naturally inharmonic circular membranes can allow them to produce quasi-harmonic vibrations, as it is the case for tablas [8-11]. In Section 3, we solve the above problem for two large classes of non-symmetric 3-strings. We find out the optimal configuration of each of these classes by determining the values of the physical parameters that minimize the sum of squares of the deviations from harmony of their first five modes [12]. Section 4 contains our concluding remarks.

2. MATHEMATICAL MODEL

To set up the mathematical framework of this problem, let x_j be the arc-length parameter for the jth string whose length at rest is l_j, $j=1,2,3$. We shall use a coordinate system where the junction point of the three strings is located at $x_j=0$, $j=1,2,3$. Therefore $0 \le x_j \le l_j$. We denote by $u^j(x_j,t)$ the function describing the transverse displacement from its rest position, at time t, of the point at arc-length position x_j on the jth string. Also, let μ_j and τ_j be the mass density and tension, respectively, of the jth string. We assume that the 3-string has a mass M at the junction point of its strings. Thus, each string has a mass density given by

$$\mu_j(x_j) = \mu_j + \frac{M}{3}\delta(x_j), \quad j=1,2,3,$$

where δ is Dirac's distribution.

Using the calculus of variations, or Newton's second law [13], it is straightforward to show that the partial differential equations describing the small-amplitude transverse vibrations of the above loaded 3-string are

$$\tau_j \frac{\partial^2 u^j(x_j,t)}{\partial x_j^2} = \mu_j(x_j)\frac{\partial^2 u^j(x_j,t)}{\partial t^2}, \quad j=1,2,3. \tag{1}$$

We also get the following condition applying to the junction point

$$\sum_{j=1}^{3} \tau_j \frac{\partial u^j(0,t)}{\partial x_j} = M\frac{\partial^2 u^k(0,t)}{\partial t^2}, \quad k=1,2,3. \tag{2}$$

Another condition to be satisfied at the junction point is the continuity of displacement of the strings, namely

$$u^1(0,t) = u^2(0,t) = u^3(0,t). \tag{3}$$

Finally, we have the boundary conditions

$$u^j(l_j,t) = 0, \quad j=1,2,3. \tag{4}$$

A normalization of the three string lengths will help us to determine the normal modes of vibration of the loaded 3-string. Let $x_j = \pi x_j / l_j$ for $j=1,2,3$. Then $0 \le x \le \pi$ parametrizes each string. Setting $v^j(x,t) = u^j(l_j x/\pi, t)$ for $j=1,2,3$, the problem formed by equations (1)-(4) becomes

$$\frac{\tau_j \pi^2}{l_j^2}\frac{\partial^2 v^j(x_j,t)}{\partial x^2} = \mu_j \frac{\partial^2 v^j(x_j,t)}{\partial t^2}, \quad 0<x<\pi,\ t>0,\ j=1,2,3, \tag{5}$$

$$\sum_{j=1}^{3}\frac{\tau_j \pi}{l_j}\frac{\partial v^j(0,t)}{\partial x} = M\frac{\partial^2 v^k(0,t)}{\partial t^2}, \quad t>0,\ k=1,2,3, \tag{6}$$

$$v^1(0,t) = v^2(0,t) = v^3(0,t),\ t \ge 0, \tag{7}$$

$$v^j(\pi,t) = 0,\ t\ge 0,\ j=1,2,3. \tag{8}$$

Let us seek for time-harmonic solutions of (5)-(8) through the method of separation of variables. To this end, we set

$$v^j(x,t) = X^j(x)T(t),\ j=1,2,3. \tag{9}$$

Replacing (9) into equation (5) yields

$$\frac{d^2X^j(x)}{dx^2}+\frac{\lambda^2 l_j^2\mu_j}{\tau_j\pi^2}X^j(x)=0 \tag{10}$$

and

$$\frac{d^2T(t)}{dt^2}+\lambda^2T(t)=0, \tag{11}$$

where $-\lambda^2$ is the separation constant. By replacing (9) into equations (6)-(8), we also get

$$\sum_{j=1}^{3}\frac{\tau_j\pi}{l_j}\frac{dX^j(0)}{dx}=-M\lambda^2X^k(0), k=1,2,3, \tag{12}$$

$$X^1(0)=X^2(0)=X^3(0), \tag{13}$$

$$X^j(\pi)=0, \quad j=1,2,3. \tag{14}$$

The solution of equation (10) is

$$X^j(x)=\mathcal{A}_j\cos\left(\frac{\lambda l_j}{c_j\pi}x\right)+\mathcal{B}_j\sin\left(\frac{\lambda l_j}{c_j\pi}x\right), \theta$$

where $\mathcal{A}_j$ and $\mathcal{B}_j$ are arbitrary constants, and $c_j=\sqrt{\tau_j/\mu_j}$. Taking (14) into account, we obtain

$$X^j(x)=A_j\sin\left(\frac{\lambda l_j}{c_j\pi}\right)(x-\pi), \tag{15}$$

where A_j are arbitrary constants. Replacing (15) into equations (12) and (13), we get

$$\sum_{j=1}^{3}A_j\frac{\tau_j}{c_j}\cos\left(\frac{\lambda l_j}{c_j}\right)=\lambda MA_k\sin\left(\frac{\lambda l_k}{c_k}\right), k=1,2,3 \tag{16}$$

and

$$A_1\sin\left(\frac{\lambda l_1}{c_1}\right)=A_2\sin\left(\frac{\lambda l_2}{c_2}\right)=A_3\sin\left(\frac{\lambda l_3}{c_3}\right). \tag{17}$$

Equations (16) and (17) form a homogeneous system of three linear equations for the unknowns A_1, A_2 and A_3, which has a non-trivial solution if and only if the determinant of its matrix of coefficients equals zero. This leads to the eigenvalue equation

$$\left[\sum_{j=1}^{3}c_j\mu_j\cos\left(\frac{\lambda l_j}{c_j}\right)\prod_{\substack{k\neq j\\k=1}}^{3}\sin\left(\frac{\lambda l_k}{c_k}\right)\right]-\lambda M\prod_{j=1}^{3}\sin\left(\frac{\lambda l_j}{c_j}\right)=0. \tag{18}$$

Equation (18) has an infinite number of roots $0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$, with $\lim_{x\to\infty}\lambda_n=\infty$ whose actual values depend on the values of M, l_j, τ_j and μ_j. These roots are simple and correspond to inharmonic frequencies of the 3-string [2].

To characterize the physical parameters of a loaded non-symmetric 3-string with maximal harmonicity now reduces to determine the set of values M, l_j, τ_j and μ_j, $j=1,2,3$, so that the terms of the sequence $\{\lambda_n/\lambda_1\}$ be as close as possible to those of the sequence $\{n\}$, $n\in N^*$.

We measure the global deviation from complete harmony of the overtones through the function

$$E(M,l_j,\tau_j,\mu_j)=\sum_{n=2}^{5}\left(\frac{\frac{\lambda_j}{\lambda_1}-n}{n}\right)^2, \tag{19}$$

where *n* represents the overtone rank. We divide each deviation from harmony by the rank of the corresponding harmonic to take into account the fact that the energy of a mode generally decreases with its order. The choice of 5 as upper bound in the summation, that is the fundamental frequency and its first four overtones, is somewhat arbitrary but remains consistent with previous studies, higher overtones being greatly damped due to the weakness of their energies [5, 6, 8-10].

3. TWO PARTICULAR CLASSES OF NON-SYMMETRIC 3-STRINGS

To be more specific, let us solve the above problem for two particular classes of non-symmetric ergonomic 3-strings. For the first class, we set

- $\mu_1=\mu_2=\mu_3=\mu,$
- $l_1=2l, l_2=l, l_3=rl$,
- $\tau_1=2\tau\cos\theta, \tau_1=\tau_3=\tau,$

where r is any positive real number and θ is half of the angle between the second and third strings at rest, the two other angles formed by the strings being equal to $\pi-\theta$ (Figure 2). Under these conditions, equation (18) gives

$$\sqrt{2\cos\theta}\sin(\gamma)\sin(\gamma r)\cos\left(\frac{2\gamma}{\sqrt{2\cos(\theta)}}\right)$$
$$+\sin(\gamma(r+1))\sin\left(\frac{2\gamma}{\sqrt{2\cos(\theta)}}\right)$$
$$-\gamma\bar{M}\sin(\gamma)\sin(\gamma r)\sin\left(\frac{2\gamma}{\sqrt{2\cos(\theta)}}\right)=0, \quad (20)$$

where we have set $\gamma=\lambda l/c$ and $\bar{M}=M/\mu l$ for $c=\sqrt{\tau/\mu}$.

For each $(r,\theta,\bar{M})$, where r ranges from 0.01 to 2 with a step size of 0.01, θ ranges from 0 to $89\pi/180$ with a step size of $\pi/180$ and $\bar{M}$ ranges from 0 to 20 with a step size of 0.05, we find the first five positive roots of (20) using Maple with the secant root-finding method with a step size of 0.02. Since $\gamma=\lambda l/c$, the sought eigenvalues λ_n are these roots multiplied by c/l.

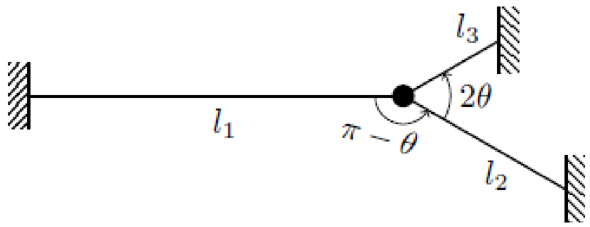

Figure 2. Schematic of a 3-string in our first class of 3-strings.

It follows from (11) that the corresponding eigenfrequencies of the non-symmetric 3-string are given by $f_n=\lambda_n/2\pi$. We then apply these eigenfrequencies to the function (19), which here reduces to

$$F(r,\theta,\bar{M})=\sum_{n=2}^{5}\left(\frac{\frac{f_n}{f_1}-n}{n}\right)^2 \quad (21)$$

Our purpose here is to determine the triplet $(r,\theta,\bar{M})$ minimizing (21). We find that this function reaches its minimal value of 0.002215 at $(r,\theta,\bar{M})=(0.56,\pi/6,1.55)$. The fundamental frequency is then given by $f_1=\gamma_1 c/2\pi l$, where $\gamma_1=1.098289$. Also, $f_n/f_1=\gamma_n/\gamma_1, n=2,3,4,5$, are respectively equal to 2.067292, 3.011902, 3.869350, 4.999067. Since the complete harmony, in relation to f_1, of the first four overtones would be to have $f_n/f_1=n$ for $n=2,3,4,5,$ we see that these overtones are very close to be harmonic.

A function associated with the optimal configuration in this set of 3-strings is

$$F^*(\theta)=\min_{r,\bar{M}}F(r,\theta,\bar{M})$$

whose graph is shown in Figure 3. This curve being smooth in a large neighborhood around $\theta=\pi/6$, a small deviation from this point will have a very small effect on the harmonicity of the corresponding configuration. The method used to examine the behavior of $F^*(\theta)$ near $\theta=\pi/6$ leads to the same result when applied to the variables r and $\bar{M}$ near $r=0.59$ and $\bar{M}=1.55$, respectively. This allows us to conclude that $(r,\theta,\bar{M})$ does not need to be exactly $(0.56,\pi/6,1.55)$ in order to obtain a configuration whose harmonicity is near its maximum.

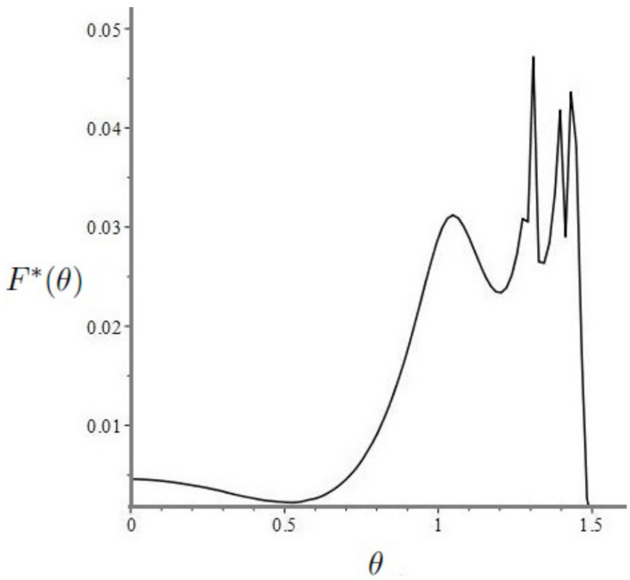

Figure 3. The graph of $F^*(\theta)$ for $0\le\theta\le 89\pi/180$

Note that the particular value of $\theta=\pi/6$ results from numerical computations, first to determine the roots of the eigenvalue equation, and second to apply the weighted least squares method to these roots. It is not possible to infer this particular value of θ through mathematical analysis or physical arguments. A more advanced numerical analysis shows that the maximum of harmonicity for other choices of l_1, l_2 and μ_i is reached at other values of θ. Figure 2 is a fair sketch of the actual optimal configuration for this class of 3-strings.

A second class of non-symmetric ergonomic 3-strings is characterized by

- $\mu_1 = \mu_2 = \mu, \mu_3 = q\mu$
- $l_1 = 2l, l_2 = l, l_3 = 6l$,
- $\tau_1 = \tau_2 = \tau_3 = \tau$,

where q is any positive real number. The three angles formed by the strings at rest are equal to $2\pi/3$ (Figure 4). Under these conditions, equa-tion (18) becomes

$$\begin{aligned}&\sin(3\gamma)\sin\left(6\gamma\sqrt{q}\right)\\&+\sqrt{q}\sin(\gamma)\sin(2\gamma)\cos(6\gamma\sqrt{q})\\&-\gamma\bar{M}\sin(\gamma)\sin(2\gamma)\sin\left(6\gamma\sqrt{q}\right)=0,\end{aligned} \tag{22}$$

where $\gamma = \lambda l/c$, $\bar{M} = M/\mu l$ and $c = \sqrt{\tau/\mu}$. For each $(q, \bar{M})$, where q ranges from 1 to 10 with a step size of 0.005, and M ranges from 0 to 2 with a step size of 0.005, we find the first five positive roots of equation (22) through the secant root-finding method with a step size of 0.02. The method was programmed with Maple, and we checked this process through the calculation of the corresponding known roots of the 3-string of the preceding example with q =1, r = 6 and $\theta = \pi/3$. As for the preceding class of non-symmetric 3-strings, the sought eigenvalues λ_n are these roots multiplied by c/l and the eigenfrequencies are given by $f_n = \lambda_n/2\pi$.

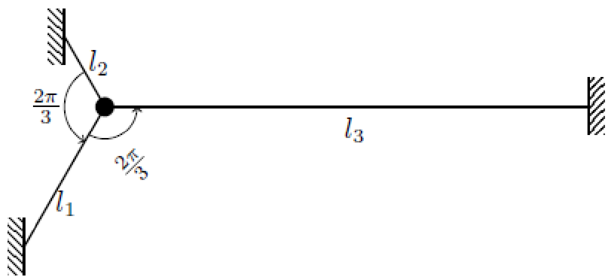

Figure 4. Schematic of a 3-string in our second class of 3-strings

We apply these eigenfrequencies to the function (19), which here reduces to

$$G(q,\bar{M}) = \sum_{n=2}^{5}\left(\frac{\frac{f_n}{f_1}-n}{n}\right)^2 \tag{23}$$

We have here to determine the pair $(q, \bar{M})$ that minimizes (23). Figure 5 is the graph of $G(q, \bar{M})$, for $0 \le G(q,\bar{M}) \le 10^{-5}$, $4 \le q \le 10$, and $0 \le \bar{M} \le 2$. This function reaches its minimal value of 9.48×10^{-9} at $(q,\bar{M}) = (6.045, 0.3696)$. For these values of q and $\bar{M}$, the fundamental frequency of the 3-string is given by $f_1 = \gamma_1 c/2\pi l$ where $\gamma_1 = 0.191658$. Moreover $f_n/f_1 = \gamma_n/\gamma_1$, n = 2, 3, 4, 5, are equal to 1.99989, 2.999899, 4.000273, 4.999872, respectively. Since the surface representing $G(q, \bar{M})$ is smooth in a region around $(q,\bar{M}) = (6.045, 0.3696)$, a small deviation from

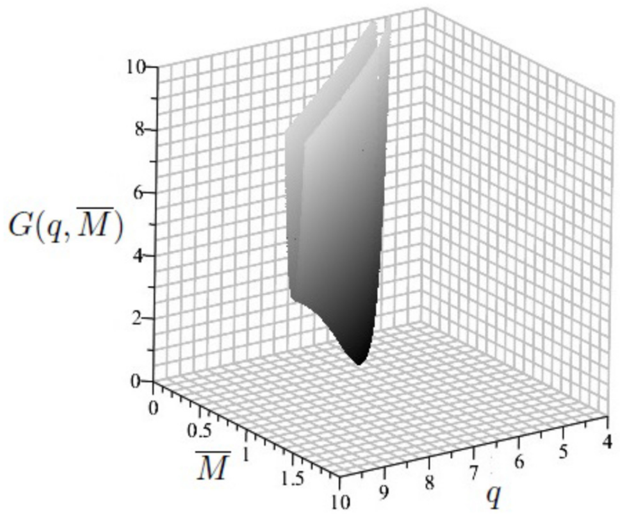

Figure 5. The graphs of $G(q,\bar{M})$) where the numbers along the vertical axis must be multiplied by 10^{-6}. This function reaches its minimal value of 9.48×10^{-9} at $(q, \bar{M}) = (6.045, 0.3696)$.

this point will have a very small effect on the harmonicity of the corresponding configuration. Again, this means that $(q, \bar{M})$ does not need to be exactly that corresponding to the minimal value of $G(q, \bar{M})$ in order to get a configuration whose harmonicity is near its maximum.

4. CONCLUDING REMARKS

By giving different values to μ in one or the other of the above best configurations, it should be possible to get a set of 3-strings having the same geometry that produce sounds with different fundamental frequencies. Musical instruments built with such a set of 3-strings should produce sounds having more musical qualities than all other 3-string configurations in their class, while still being able to create inharmonic sound effects when wanted.

The sounds of a 3-string having the optimal configuration of our second class will be much more harmonic than those produced by a 3-string having the optimal configuration of our first class. Notice that the fundamental frequency of vibrations of the 3-strings with maximal harmonicity in the first class is $f_1 = 1.098289c/2\pi l$, whereas that of the second class is $f_1 = 0.191658c/2\pi l$. This means that musical instruments built with 3-strings having the optimal configuration of the second example will generally produce sounds having frequencies lower than sounds produced by 3-strings based on the first example, when the parameters of these sets of 3-strings yield the same values for *c* and *l*. Therefore, if we put regular guitars in correspondence with musical instruments built with our first example optimal configuration of 3-strings, then those associated with our second example could correspond to bass guitars.

REFERENCES

[1] Kline M., *Mathematical Thought from Ancient to Modern Times*, Oxford, New York, 1972, pp. 503-522.

[2] Perov P., Johnson W., Perova-Mello N., The physics of guitar string vibrations, *American. Journal of Physics,* Vol. 84, 2016, pp. 38-43.

[3] Hayashi T., Azegami H., Shape optimization of bodies of musical instruments, *JSIAM Letters,* Vol. 10, 2018, pp. 29- 32.

[4] Schmidt E. J. P. G., On the modelling and exact controllability of networks of vibrating strings, SIAM Journal on Control and Optimization, Vol. 30, 1992, pp 229-245.

[5] Gaudet S., Gauthier C., LeBlanc V.G., On the vibrations of an N-string, *Journal of Sound and Vibration*, Vol. 238, 2000, pp. 147-169.

[6] Gaudet S., Gauthier C., Léger S., C. Walker C., The vibrations of a real 3-string: the timbre of the tritare, *Journal of Sound and Vibration*, Vol. 281, 2005, pp. 219-234.

[7] Bolwell J.E., On Rayleigh's equations for the vibrations of a loaded flexible string, *European Journal of Physics*, Vol 20, 1999, pp. 305-312.

[8] Raman C.V., The Indian musical drums, *Proceedings of the Indian Academy of Sciences*, Vol. A1, 1934, pp. 179-188.

[9] Fletcher N.H., Rossing T.D., *The Physics of Musical Instruments, Second Edition,* Springer-Verlag, New York, 1998, pp. 609-615.

[10] Gaudet S., Gauthier C., Léger S., The evolution of harmonic Indian musical drums: A mathematical perspective, *Journal of Sound and Vibration*, Vol. 291, 2006, pp. 388- 394.

[11] Sathej G., Adhikari R., The eigenspectra of Indian musical drums, *The Journal of Acoustical Society of America*, Vol. 125, 2009, pp. 831-838.

[12] Tazeroualti M., Designing a progressive lens, *2nd International Conference on Curves and Surfaces in Geometric Design*, A.K. Peters, Natick, MA, 1994, pp. 467-474.

[13] Gomez B.J., Repetto C.E., Stia C.R., Welti R., Oscillations of a string with concentrated masses, *European Journal of Physics*, Vol. 28, 2007, pp. 961-975.