On the Dynamics with s Collisions of a Spatial Physical Pendulum

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Abstract: - This paper discusses the dynamics of a general rigid solid, hanged by a spring excited at its other end. The approach is a multibody one and the matrix equation of motion is obtained from the general matrix differential equation, by canceling the matrix of constraints. The motion of the rigid solid is limited to a zone in the space, the borders being two vertical walls. A great attention is paid to the apparition of a frictionless collision because the equations of motion of the rigid solid must be translated other system of coordinates. An example highlights the theory.

Keywords: - multibody approach, frictionless collisions, equation of motion, different reference frames

1. INTRODUCTION

The study of the dynamics of a classical pendulum is an old and still researched problem [1]. Usually, authors consider a mathematical pendulum, hanged or not by different elastic elements and with different excitations [1, 2]. The application of such pendulum for the control of the behavior of a building is described in [3].

Some aspects concerning of the resonance of a spherical physical pendulum hanged by a linear spring, damped and excited by a rotational torque are presented in [4].

Moreover, the classical mathematical pendulum is presented as example in almost any book of mechanics, some of them discussing the problem by a multibody approach [5–18]. The physical pendulum is not so often discussed because the spatial dynamics involves many difficulties.

The problem of the dynamics of a rigid solid with general constraints is generally presented in [19]. Vanishing the matrix of constraints one may obtain the matrix differential equation of the rigid solid. In this form, the obtained equations are equivalent to those obtained from the Lagrange second order equations, but they can be more quickly written.

The collision problem between a rigid solid and an obstacle can be studied using the equations developed in [20]. The equations are directly written in matrix form.

Using the symbolism from references [19] and [20], the dynamics of pendulum is treated in a multibody approach.

2. PROBLEM FORMULATION

We consider (Figure 1) a rigid solid of mass m hanged by a spring AP of stiffness k. The motions of the point P along the axes O_0X, O_0Y and O_0Z are known and given by $u_X(t)$, $u_Y(t)$ and $u_Z(t)$, where t is the time.

Attached to the rigid solid is the reference mobile system $Cxyz$, relative to which one knows the moments of inertia.

The center of weight of the rigid solid is denoted by C and it has the coordinates X_C, Y_C, Z_C relative to the fixed reference system.

At the distances d_1 and d_2 with respect to the plane O_0XZ there ate two walls of infinite length. These walls may be collided without friction by the rigid solid, that is, the motion of the rigid solid has to be situated between the two walls.

One asks for the determination of motion of the rigid solid.

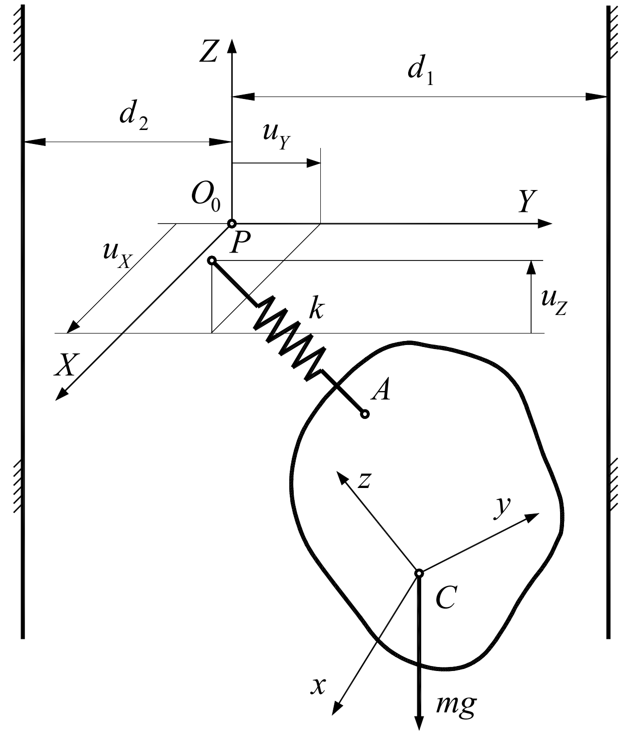

Figure 1. Problem formulation

3. MOTION WITHOUT COLLISION

3.1. Notations

We will use the following notations:

– the rotational angles ψ, θ, φ that correspond to the rotations about the x, y, z axes;

– the matrices

$$[\boldsymbol{\psi}]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\psi & -\sin\psi\\ 0 & \sin\psi & \cos\psi\end{bmatrix},\quad [\boldsymbol{\theta}]=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix},\quad [\boldsymbol{\varphi}]=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}; \tag{1}$$

– the rotational matrix

$$[\mathbf{A}]=[\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}]=\begin{bmatrix}c\theta c\varphi & -c\theta s\varphi & s\theta\\ s\psi s\theta c\varphi+c\psi s\varphi & -s\psi s\theta s\varphi+c\psi c\varphi & -s\psi c\theta\\ -c\psi\psi\theta c\varphi+s\psi s\varphi & c\psi s\theta s\varphi+s\psi c\varphi & c\psi c\theta\end{bmatrix}; \tag{2}$$

– the column matrices

$$\{\mathbf{u}_\psi\}=[1\quad 0\quad 0]^{\mathrm{T}},\ \{\mathbf{u}_\theta\}=[0\quad 1\quad 0]^{\mathrm{T}},\ \{\mathbf{u}_\varphi\}=[0\quad 0\quad 1]^{\mathrm{T}}; \tag{3}$$

– the square matrix $[\mathbf{Q}]$

$$[\mathbf{Q}]=[\boldsymbol{\varphi}]^{\mathrm{T}}\left[[\boldsymbol{\theta}]^{\mathrm{T}}\{\mathbf{u}_\psi\}\quad \{\mathbf{u}_\theta\}\quad \{\mathbf{u}_\varphi\}\right]=\begin{bmatrix}\cos\varphi\cos\theta & \sin\varphi & 0\\ -\sin\varphi\cos\theta & \cos\varphi & 0\\ \sin\theta & 0 & 1\end{bmatrix}; \tag{4}$$

– the matrices for the angular velocity

$$\{\boldsymbol{\omega}\}=[\mathbf{Q}]\begin{bmatrix}\dot\psi\\ \dot\theta\\ \dot\varphi\end{bmatrix}=\begin{bmatrix}\dot\psi\cos\varphi\cos\theta+\dot\theta\sin\varphi\\ -\dot\psi\sin\varphi\cos\theta+\dot\theta\cos\varphi\\ \dot\psi\sin\theta+\dot\varphi\end{bmatrix}=\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}, \tag{5}$$

$$[\boldsymbol{\omega}]=\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}; \tag{6}$$

– the square matrix $[\mathbf{S}]$

$$[\mathbf{S}]=\begin{bmatrix}0 & -mz_C & my_C\\ mz_C & 0 & -mx_C\\ -my_C & mx_C & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}; \tag{7}$$

– the matrix of the central principal moments of inertia

$$[\mathbf{J}_C]=\begin{bmatrix}J_x & 0 & 0\\ 0 & J_y & 0\\ 0 & 0 & J_z\end{bmatrix}; \tag{8}$$

– the square matrix $[\mathbf{m}]$

$$[\mathbf{m}]=\begin{bmatrix}m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & m\end{bmatrix}; \tag{9}$$

– the square matrix $[\mathbf{r}_A]$

$$[\mathbf{r}_A]=\begin{bmatrix} 0 & -z_A & y_A \\ z_A & 0 & -x_A \\ -y_A & x_A & 0 \end{bmatrix}; \tag{10}$$

– the column matrix of the coordinates of the point A relative to the mobile reference system

$$\{\mathbf{r}_A\}=[x_A \quad y_A \quad z_A]^{\mathrm{T}}; \tag{11}$$

– the column matrix of the coordinates of the point A relative to the fixed reference system

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}=\begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}+[\mathbf{A}]\{\mathbf{r}_A\}; \tag{12}$$

– the column matrix of the coordinates of the point P relative to the fixed reference system

$$\begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix}=\begin{bmatrix} u_X(t) \\ u_Y(t) \\ u_Z(t) \end{bmatrix}; \tag{13}$$

– the column matrix of the generalized forces corresponding to the elastic force in the spring

$$\{\mathbf{F}\}_1=\frac{k(AP-l_0)}{AP}\begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} X_P-X_A \\ X_P-Y_A \\ X_P-Z_A \end{bmatrix}; \tag{14}$$

– the column matrix of the generalized forces corresponding to the weight

$$\{\mathbf{F}\}_2=[0 \quad 0 \quad -mg \quad 0 \quad 0 \quad 0]^{\mathrm{T}}; \tag{15}$$

– the square matrices

$$[\mathbf{U}_\psi]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ [\mathbf{U}_\theta]=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ [\mathbf{U}_\varphi]=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{16}$$

– the square matrix of partial derivatives

$$[\boldsymbol{\psi}_p]=[\mathbf{U}_\psi][\boldsymbol{\psi}],\ [\boldsymbol{\theta}_p]=[\mathbf{U}_\theta][\boldsymbol{\theta}],\ [\boldsymbol{\varphi}_p]=[\mathbf{U}_\varphi][\boldsymbol{\varphi}]; \tag{17}$$

– the matrices of the partial derivatives of the rotational matrix

$$[\mathbf{A}_\psi]=[\boldsymbol{\psi}_p][\boldsymbol{\theta}][\boldsymbol{\varphi}],\ [\mathbf{A}_\theta]=[\boldsymbol{\psi}][\boldsymbol{\theta}_p][\boldsymbol{\varphi}],\ [\mathbf{A}_\varphi]=[\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}_p]; \tag{18}$$

– the derivative of the rotational matrix with respect to time

$$[\dot{\mathbf{A}}]=\dot{\psi}[\mathbf{A}_\psi]+\dot{\boldsymbol{\theta}}[\mathbf{A}_\theta]+\dot{\boldsymbol{\varphi}}[\mathbf{A}_\varphi]; \tag{19}$$

– the matrices

$$\{\mathbf{s}\}=[X_C \quad Y_C \quad Z_C]^{\mathrm{T}},\ \{\boldsymbol{\beta}\}=[\psi \quad \theta \quad \varphi]^{\mathrm{T}},. \tag{20}$$

$$\{\mathbf{q}\}=\left[\{\mathbf{s}\}^{\mathrm{T}} \quad \{\boldsymbol{\beta}\}^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{21}$$

3.2. Matrix differential equation of motion

One successively calculates

$$\{\tilde{\mathbf{F}}_{\mathbf{s}}\}=-[[\mathbf{A}][\mathbf{S}]^{\mathrm{T}}[\dot{\mathbf{Q}}]+[\dot{\mathbf{A}}][\mathbf{S}]^{\mathrm{T}}[\mathbf{Q}]]\{\dot{\boldsymbol{\beta}}\} =[0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{22}$$

$$[\mathbf{Q}_\varphi]=\begin{bmatrix} \cos\varphi\sin\theta & -\sin\varphi & 0 \\ -\sin\varphi\sin\theta & -\cos\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

$$[\mathbf{Q}_\theta]=\begin{bmatrix} \sin\varphi\cos\theta & 0 & 0 \\ \cos\varphi\cos\theta & 0 & 0 \\ -\sin\theta & 0 & 0 \end{bmatrix}, \tag{24}$$

$$[\dot{\mathbf{Q}}]=\dot{\varphi}[\mathbf{Q}_\varphi]+\dot{\theta}[\mathbf{Q}_\theta], \tag{25}$$

$$\{\tilde{\mathbf{F}}_{\boldsymbol{\beta}}\}=-[[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\dot{\mathbf{Q}}]+[\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\omega}][\mathbf{J}_C][\mathbf{Q}]]\{\dot{\boldsymbol{\beta}}\}, \tag{26}$$

$$\{\tilde{\mathbf{F}}_{\mathbf{q}}\}=\left[\{\tilde{\mathbf{F}}_{\mathbf{s}}\}^{\mathrm{T}}\{\tilde{\mathbf{F}}_{\boldsymbol{\beta}}\}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{27}$$

$$\{\tilde{\mathbf{F}}_{\mathbf{q}}\}=\{\mathbf{F}\}_1+\{\mathbf{F}\}_2, \tag{28}$$

$$[\mathbf{M}]=\begin{bmatrix}[\mathbf{m}] & [\mathbf{A}][\mathbf{S}]^{\mathrm{T}}[\mathbf{Q}]\\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{S}][\mathbf{A}]^{\mathrm{T}} & [\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}]\end{bmatrix}= =\begin{bmatrix}[\mathbf{m}] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}]\end{bmatrix}. \qquad (29)$$

The matrix differential equation of motion reads

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\}=\{\mathbf{F}_{\mathbf{q}}\}+\{\widetilde{\mathbf{F}}_{\mathbf{q}}\}, \qquad (30)$$

which is the simplified form of that reported in [19] when the matrix of constraints vanishes.

4. COLLISION AT ONE POINT

Let be $\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ the normal at the contact point P_1 and let $\mathbf{P}$ be the impulse at the same point.

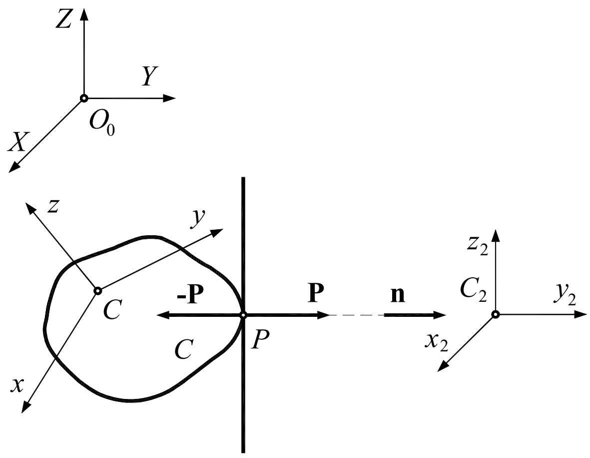

Figure 2. Collision

We calculate (Fig. 2):
– the cross product

$$\mathbf{CP}_1\times\mathbf{n}=d\mathbf{i}+e\mathbf{j}+f\mathbf{k}\ ; \qquad (31)$$

– the column matrices

$$\{\mathbf{N}_1\}=\begin{bmatrix}a & b & c & d & e & f\end{bmatrix}^{\mathrm{T}},\quad \{\mathbf{N}_2\}=\begin{bmatrix}0 & -1 & 0 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}; \qquad (32)$$

– the square matrix

$$[\boldsymbol{\eta}]=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{bmatrix}; \qquad (33)$$

– the matrices of inertia

$$[\mathbf{M}_1]=\begin{bmatrix}0&0&0&m&0&0\\0&0&0&0&m&0\\0&0&0&0&0&m\\J_x&0&0&0&0&0\\0&J_y&0&0&0&0\\0&0&J_z&0&0&0\end{bmatrix},\quad [\mathbf{M}_1]^{-1}=\begin{bmatrix}0&0&0&J_x^{-1}&0&0\\0&0&0&0&J_y^{-1}&0\\0&0&0&0&0&J_z^{-1}\\m^{-1}&0&0&0&0&0\\0&m^{-1}&0&0&0&0\\0&0&m^{-1}&0&0&0\end{bmatrix} \qquad (34)$$

and

$$[\mathbf{M}_2]=\begin{bmatrix}0&0&0&\infty&0&0\\0&0&0&0&\infty&0\\0&0&0&0&0&\infty\\\infty&0&0&0&0&0\\0&\infty&0&0&0&0\\0&0&\infty&0&0&0\end{bmatrix},\quad [\mathbf{M}_2]^{-1}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}; \qquad (35)$$

– the velocities of the centers of weight relative to the fixed reference frame

$$\mathbf{v}^*_{C_1}=\dot{X}_C\mathbf{i}_0+\dot{Y}_C\mathbf{j}_0+\dot{Z}_C\mathbf{k}_0,\quad \{\mathbf{v}^*_{C_1}\}=\begin{bmatrix}\dot{X}_C & \dot{Y}_C & \dot{Z}_C\end{bmatrix}^{\mathrm{T}}, \qquad (36)$$

$$\mathbf{v}^*_{C_2}=\mathbf{0},\ \{\mathbf{v}^*_{C_2}\}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}^{\mathrm{T}}; \qquad (37)$$

– the velocities of the centers of weight relative to the mobile reference systems

$$\{\mathbf{v}_{C_1}\}=[\mathbf{A}]^{\mathrm{T}}\{\mathbf{v}^*_{C_1}\},\ \{\mathbf{v}_{C_2}\}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}^{\mathrm{T}}; \qquad (38)$$

– the inertances [20]

$$g_1 = \{\mathbf{N}_1\}^{\mathrm{T}} [\boldsymbol{\eta}][\mathbf{M}_1]^{-1} \{\mathbf{N}_1\} = \frac{1}{m} + \frac{d^2}{J_x} + \frac{e^2}{J_y} + \frac{f^2}{J_z} \quad (39)$$

,

$$g_2 = \{\mathbf{N}_2\}^{\mathrm{T}} [\boldsymbol{\eta}][\mathbf{M}_2]^{-1} \{\mathbf{N}_2\} = 0 ; \quad (40)$$

– the matrices of velocities before collision

$$\{\mathbf{v}_1^0\} = [\omega_x^0 \quad \omega_y^0 \quad \omega_z^0 \quad v_{1x}^0 \quad v_{1y}^0 \quad v_{1z}^0]^{\mathrm{T}}, \quad (41)$$

$$[v_{1x}^0 \quad v_{1y}^0 \quad v_{1z}^0]^{\mathrm{T}} = [\mathbf{A}]^{\mathrm{T}} [\dot{X}_C \quad \dot{Y}_C \quad \dot{Z}_C]^{\mathrm{T}}, \quad (42)$$

$$\{\mathbf{v}_2^0\} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} ; \quad (43)$$

– the velocities v_{1n}^0, v_{2n}^0

$$v_{1n}^0 = \{\mathbf{N}_1\}^{\mathrm{T}} [\boldsymbol{\eta}] \{v_1^0\},\ v_{2n}^0 = \{\mathbf{N}_2\}^{\mathrm{T}} [\boldsymbol{\eta}] \{v_2^0\} = 0 ; \quad (44)$$

– the velocity v_{12n}^0

$$v_{12n}^0 = v_{1n}^0 - v_{2n}^0 = v_{1n}^0 ; \quad (45)$$

– the impulse [20]

$$P = \frac{(1+k) v_{12n}^0}{g_1 + g_2} = \frac{(1+k) v_{1n}^0}{g_1}, \quad (46)$$

where k is the coefficient of restitution;

– the velocities after collision relative to mobile axes [20]

$$\begin{aligned} \{\mathbf{v}_1\} &= \{\mathbf{v}_1^0\} - P [\mathbf{M}_1]^{-1} \{\mathbf{N}_1\}, \\ \{\mathbf{v}_2\} &= \{\mathbf{v}_2^0\} + P [\mathbf{M}_2]^{-1} \{\mathbf{N}_2\} \\ &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} ; \end{aligned} \quad (47)$$

– the velocities after collision relative to fixed frame

$$\begin{aligned} \{\mathbf{v}_{C_1}^{**}\} &= [\mathbf{A}] \{v_{1x} \quad v_{1y} \quad v_{1z}\}^T, \\ \{\mathbf{v}_{C_2}^{**}\} &= [0 \quad 0 \quad 0]^T. \end{aligned} \quad (48)$$

When a collision occurs one has to modify the velocities of the rigid solid according to the previous formulae. The time remains unchanged.

Let us observe that the existence of a collision is characterized by the fact that the velocity of the point P_1 is orientated to the wall and the distance between the point P_1 and the wall is equal to zero. Both conditions must hold true. For instance, in Fig. 2, the velocity of the point P must have a component onto the direction of the normal $\mathbf{n}$, and that component must have the sense of the normal.

5. NUMERICAL EXAMPLE

As an example we consider a ball (Fig. 3) of mass m and radius r for which the reference frames are considered as in Figure.

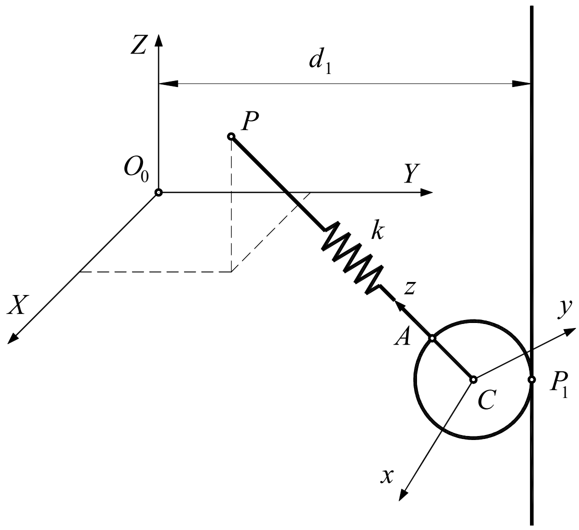

Figure 3. Numerical example

The suspension point A has the coordinates $x_A = 0$, $y_A = 0$, $z_A = r$.

Let P_1 be an arbitrary point of the ball. From the relation

$$\begin{bmatrix} X_{P_1} \\ Y_{P_1} \\ Z_{P_1} \end{bmatrix} = \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} + [\mathbf{A}] \begin{bmatrix} x_{P_1} \\ y_{P_1} \\ z_{P_1} \end{bmatrix}, , \quad (49)$$

considering that the point P_1 is the collision point between the ball and the vertical wall situated at distance d_1, that is, $X_{P_1} = X_C$, $Y_{P_1} = Y_C + r = d_1$, $Z_{P_1} = Z_C$, one gets

$$\begin{bmatrix} X_C \\ d_1 \\ Y_C \end{bmatrix} = \begin{bmatrix} X_C \\ d_1 - r \\ Y_C \end{bmatrix} + [\mathbf{A}] \begin{bmatrix} x_{P_1} \\ y_{P_1} \\ z_{P_1} \end{bmatrix}, \quad (50)$$

wherefrom

$$\begin{bmatrix} x_{P_1} \\ y_{P_1} \\ z_{P_1} \end{bmatrix} = [\mathbf{A}]^{\mathrm{T}} \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}. \quad (51)$$

Denoting by P_2 the collision point between the ball and the vertical wall situated at distance d_2, for which $X_{P_2} = X_C$, $Y_{P_2} = Y_C - r = -d_2$, $Z_{P_2} = Z_C$, it results

$$\begin{bmatrix} x_{P_2} \\ y_{P_2} \\ z_{P_2} \end{bmatrix} = [\mathbf{A}]^{\mathrm{T}} \begin{bmatrix} 0 \\ -r \\ 0 \end{bmatrix}. \quad (52)$$

We have

$$[\mathbf{J}_C] = \frac{2mr^2}{5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (53)$$

$$[\mathbf{r}_A] = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \{\mathbf{r}_A\} = [0 \quad 0 \quad r]^{\mathrm{T}}, \quad (54)$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}} = r\begin{bmatrix} 0 & -\cos\psi\cos\theta & -\sin\psi\cos\theta \\ \cos\theta & \sin\psi\sin\theta & -\sin\psi\sin\theta \\ 0 & 0 & 0 \end{bmatrix}, \quad (55)$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}] = \frac{2mr^2}{5} \begin{bmatrix} 1 & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & 1 \end{bmatrix}. \quad (55)$$

As numerical values, we consider the following: $r = 0.05\,\mathrm{m}$, $m = 4.0\,\mathrm{kg}$, $a_x = 0.15\,\mathrm{m}$, $a_y = 0.15\,\mathrm{m}$, $a_z = 0.10\,\mathrm{m}$ (amplitudes of u_X, u_Y, u_Z), $\omega_x = 1.0\,\mathrm{s}^{-1}$, $\omega_y = 1.0\,\mathrm{s}^{-1}$, $\omega_z = 1.0\,\mathrm{s}^{-1}$ (pulsations of u_X, u_Y, u_Z), $k = 500\,\mathrm{N/m}$ (stiffness of the spring), $l = 0.5\,\mathrm{m}$ (the length of the spring at rest), $d_1 = 0.12\,\mathrm{m}$, $d_2 = 0.10\,\mathrm{m}$, $k_1 = 0.7$, $k_2 = 0.6$ (the coefficients of restitution), $g = 9.8065\,\mathrm{m/s^2}$ (the gravitational acceleration). The duration of simulation is $t_{\max} = 5\,\mathrm{s}$, while the step of the simulation is $\Delta t = 0.001\,\mathrm{s}$. We will call these values as the standard ones.

The initial values are: $X_C^0 = a_x = 0.15\,\mathrm{m}$, $Y_C^0 = 0\,\mathrm{m}$, $Z_C^0 = a_z - l - r = -0.45\,\mathrm{m}$, $\psi^0 = 0\,\mathrm{rad}$, $\theta^0 = 0\,\mathrm{rad}$, $\varphi^0 = 0\,\mathrm{rad}$, $\dot{X}_C^0 = 0\,\mathrm{m/s}$, $\dot{Y}_C = 0\,\mathrm{m/s}$, $\dot{Z}_C^0 = 0\,\mathrm{m/s}$, $\dot{\psi}^0 = 0\,\mathrm{rad/s}$, $\dot{\theta}^0 = 0\,\mathrm{rad/s}$, $\dot{\varphi}^0 = 0\,\mathrm{rad/s}$.

The oscillations of the point P are defined by

$$u_X = a_x\cos(\omega_x t),\ u_Y = a_y\sin(\omega_y t),\ u_Z = a_z[\cos(\omega_z t) + \sin(\omega_z t)]. \quad (56)$$

We also consider the variants:

i) $k_1 = 1.0$, $k_2 = 1.0$;

ii) $k_1 = 0.0$; $k_2 = 0.0$;

iii) $\omega_x = 2.0\,\mathrm{s}^{-1}$, $\omega_y = 3.0\,\mathrm{s}^{-1}$, $\omega_z = 4.0\,\mathrm{s}^{-1}$;

iv) $t_{\max} = 15\,\mathrm{s}$;

v) $d_1 = 0.12\,\mathrm{m}$, $d_2 = 0.12\,\mathrm{m}$, $k_1 = 1.0$, $k_2 = 1.0$.

The results of the simulation are given in the next figures.

We denote with t_1 the time at which appears the first collision with the first wall, with t_2 the time at which appears the last collision with the first wall, with t_3 the time at which appears the first collision with the second wall, with t_4 the time at which appears the last collision with the second wall, with N_1 the number of collision with the first wall, with N_2 the number of collisions with the second wall, with P_1 the impulse at the first collision with the first wall, with P_2 the impulse at the last collision with the first wall, with P_3 the impulse at the first collision with the second wall, and with P_4 the impulse at the last collision with the second wall. These values are presented in Table 1 (for the first wall) and Table 2 (for the second wall).

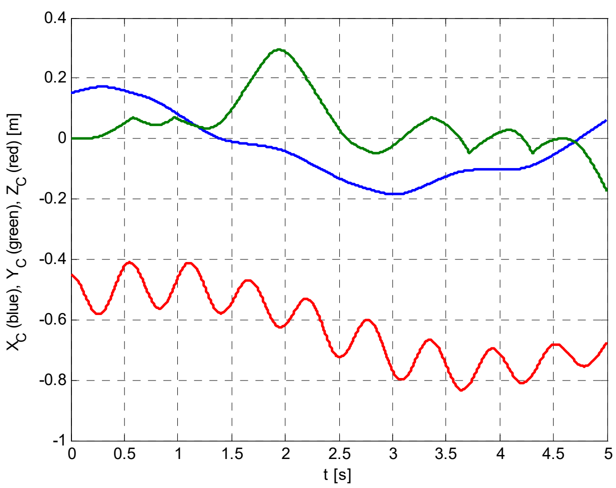

Figure 4. Time histories $X_C = X_C(t)$, $Y_C = Y_C(t)$, $Z_C = Z_C(t)$, standard variant

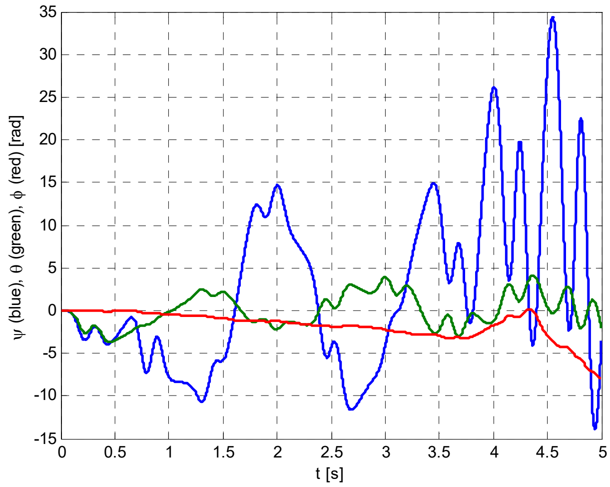

Figure 5. Time histories $\psi = \psi(t)$, $\theta = \theta(t)$, $\varphi = \varphi(t)$, standard variant

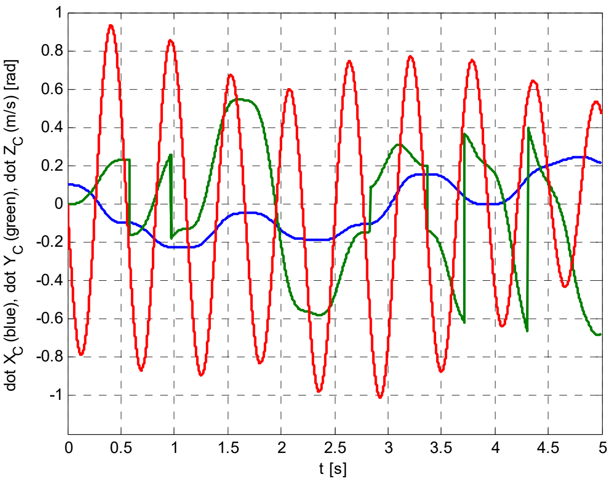

Figure 6. Time histories $\dot{X}_C = \dot{X}_C(t)$, $\dot{Y}_C = \dot{Y}_C(t)$, $\dot{Z}_C = \dot{Z}_C(t)$, standard variant

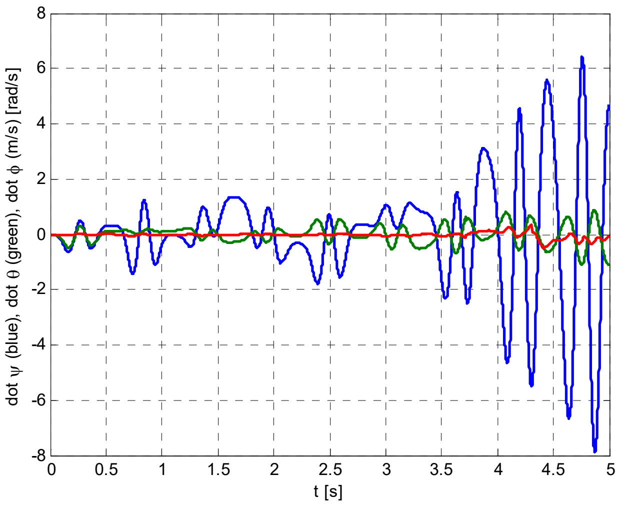

Figure 7. Time histories $\dot{\psi} = \dot{\psi}(t)$, $\dot{\theta} = \dot{\theta}(t)$, $\dot{\varphi} = \dot{\varphi}(t)$, standard variant

Analyzing the results presented in the Figures and Tables, one may state:

– increasing the time of simulation, the numbers of collisions increase;

– modifying the values of ω_x, ω_y and ω_z, one may change the times at which appear the collisions;

– the numbers of collisions at each wall, obtained for the variant ii), must be seen circumspectly. In fact, the numbers of collisions are smaller, the ball remaining pasting to the corresponding wall. For instance, for the first wall, at the point P_1, the velocity after the collision becomes null onto the direction of normal at a given moment of iteration. At the new moment that differs from the previous by the step Δt, the forces which act upon the ball (the elastic force in the spring and the weight of the ball) lead to a new velocity of the point P_1 for which the component along the normal has positive value. The situation is similar for the point P_2 where appear the collisions with the second wall;

– symmetric system (as in the case v)) do not necessary lead to equal values for the number of collisions at the two walls. These numbers tend to equal limits for sufficiently large time of simulation.

Table 1. Some characteristic of the collisions with the first wall

Variant	t_1	t_2	P_1	P_2	N_1
Standard	0.587	3.371	1.5662	1.3345	3
V1	0.587	2.993	1.8426	1.0106	2
V2	0.587	0.788	0.9213	0.0039	202
V3	0.390	0.959	3.1385	2.9675	2
V4	0.587	8.734	1.5662	1.0181	9
V5	0.587	2.993	1.8426	1.011	2

Table 2. Some characteristic of the collisions with the second wall

Variant	t_3	t_4	P_3	P_4	N_2
Standard	2.84	4.309	0.9041	4.2634	3
V1	3.393	3.393	2.8743	2.8743	1
V2	3.572	4.126	1.0312	0.0810	555
V3	-	-	-	-	0
V4	2.840	11.689	0.9041	0.0050	1128
V5	3.449	3.449	2.8387	3.449	1

6. CONCLUSIONS

This paper presents a new method for the study of a spherical physical pendulum. The method described in this paper is based on a multibody approach and it directly offers the study of the motion of such a pendulum without any collision. The appearance of the collisions complicates the problem because the study of the collision is performed in another reference system.

The problem of the collision with friction is more complicated and it requires a different approach. Let us observe that even in the case of frictionless collision it is possible to obtain that the velocity at one collision point is tangent to the obstacle, that is, a collision with friction. In all these situations we neglected the friction.

In a future work we will discuss the case in which the rigid body has constraints.

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