On the Dynamics of the Constrained Rigid Solid Acted by Conservative Forces. Part II: Examples

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Abstract: - This paper is the second part of our previous one. We present here two examples which highlight the theory. The first example is a spatial one, the conservative forces being the weight and linear elastic forces. The second example is a planar one and it involves as conservative forces the weight, a linear elastic force and a non-linear elastic force. Different constraints are considered and the calculation of the corresponding matrices of constraints is detailed. We present the main steps in solving the problems and discuss different way in approaching of the solutions.

Keywords: - multibody approach, conservative forces, equation of motion, equilibrium

1. INTRODUCTION

This paper presents some examples in order to highlight the theory described in [16]. The first example is a spatial one and involves a redundant degree of freedom. The second one is a planar one and the solution is obtained either from the general spatial one, or from the equations deduced in [16]. The conservative forces are given by the weight and by springs (which are not necessary linear as one may see in the case of planar example).

We consider that the constraints are described by the conditions that some points of the rigid solid has to be situated on fixed curves or surfaces. The calculations of the matrices of constraints are also presented.

The problem of the elimination of Lagrange multipliers is often discussed in the literature. An excellent presentation is given in [17]. Other variants of this problem treated in a multibody approach are presented in [15, 16].

2. EXAMPLE 1

The homogeneous bar AB of length $2l$, center of weight C and mass m is constrained (Fig. 1) to move its end B on the straight line (d) of equations

$$X = 0\,,\ Y = 2l\,. \tag{1}$$

The bar is linked to three linear elastic springs. The first spring has the stiffness k_1, connects the points A and $P_1(0,0,l)$, and its non-deformed length is l. The second spring connects the points C and $P_2(l,l,l)$ having the stiffness $k_2 = 2k$ and the non-deformed length l, while the third spring connects the points A and $P_3(0,2l,2l)$, has the stiffness $k_3 = k$ and the non-deformed length $2l$.

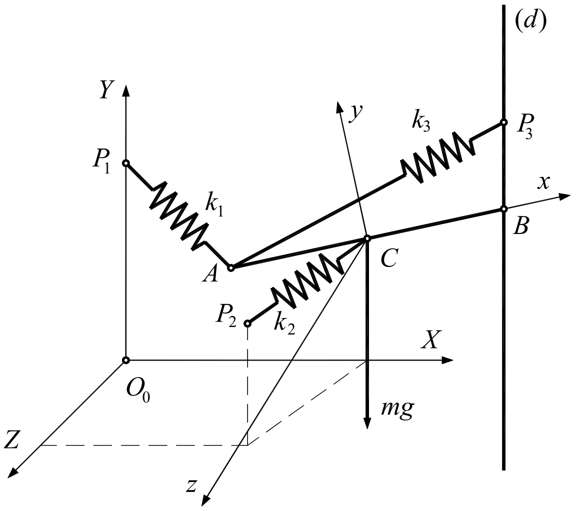

Figure 1. Example 1.

The potential V for an arbitrary spring is

$$V = -\frac{k}{2}\left(l^* - l_0\right), \tag{2}$$

where l^* is the length of the deformed spring, while l_0 is the length of the spring at rest.

Assuming that the spring connects the points Q_1 and Q_2 (point Q_1 belonging to the rigid solid), one may write

$$\begin{gathered} Q_1Q_2 = l^* = \\ \sqrt{\left(X_{Q_2} - X_{Q_1}\right)^2 + \left(Y_{Q_2} - Y_{Q_1}\right)^2 + \left(Z_{Q_2} - Z_{Q_1}\right)^2}, \end{gathered} \tag{3}$$

the components of the elastic force reading

$$\begin{aligned} F_X &= k\left(Q_1Q_2 - l_0\right)\frac{X_{Q_2} - X_{Q_1}}{Q_1Q_2}, \\ F_Y &= k\left(Q_1Q_2 - l_0\right)\frac{Y_{Q_2} - Y_{Q_1}}{Q_1Q_2}, \\ F_Z &= k\left(Q_1Q_2 - l_0\right)\frac{Z_{Q_2} - Z_{Q_1}}{Q_1Q_2}. \end{aligned} \tag{4}$$

It results

$$\begin{gathered} \{\mathbf{F}\} = \\ \frac{k^*\left(Q_1Q_2 - l_0\right)}{Q_1Q_2}\begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^{\mathrm{T}}\left[\mathbf{r}_{Q_1}\right][\mathbf{A}]^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} X_{Q_2} - X_{Q_1} \\ Y_{Q_2} - Y_{Q_1} \\ Z_{Q_2} - Z_{Q_1} \end{bmatrix}, \end{gathered} \tag{5}$$

where k^* is the stiffness of the generic spring.

Let X_C, Y_C and Z_C be the coordinates of the point C relative to the fixed frame O_0XYZ, and let ψ, θ and φ be the rotational angles according to the Bryan rotational schema. We have

$$\begin{gathered} [\boldsymbol{\psi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}, \\ [\boldsymbol{\theta}] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \\ [\boldsymbol{\varphi}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{gathered} \tag{6}$$

$$\begin{gathered} [\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}] = \\ \begin{bmatrix} c\theta c\varphi & -c\theta s\varphi & s\theta \\ s\psi s\theta c\varphi + c\psi s\varphi & -s\psi s\theta s\varphi + c\psi c\varphi & -s\psi c\theta \\ -c\psi s\theta c\varphi + s\psi s\varphi & c\psi s\theta s\varphi + s\psi c\varphi & c\psi c\theta \end{bmatrix}. \end{gathered} \tag{7}$$

The coordinates of the point B are $x_B = l$, $y_B = 0$, $z_B = 0$ (relative to the mobile reference system) resulting

$$\begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} + [\mathbf{A}]\begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}, \tag{8}$$

that is

$$\begin{gathered} X_B = X_C + l\cos\psi, \\ Y_B = Y_C + l\left(\sin\psi\sin\theta\cos\varphi + \cos\psi\sin\varphi\right), \\ Z_B = Z_C + l\left(-\cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi\right). \end{gathered} \tag{9}$$

Since the point B has to move along the straight line (d), it results the constrain functions

$$\begin{gathered} f_1\left(X_C, Y_C, Z_C, \psi, \theta, \varphi\right) = X_B = \\ X_C + l\cos\theta\cos\varphi = 0, \\ f_2\left(X_C, Y_C, Z_C, \psi, \theta, \varphi\right) = Y_B - 2l = \\ Y_C + l\left(\sin\psi\sin\theta\cos\varphi + \cos\psi\sin\varphi - 2\right) = 0, \end{gathered} \tag{10}$$

wherefrom

$$\frac{\partial f_1}{\partial X_C}=1,\ \frac{\partial f_1}{\partial Y_C}=0,\ \frac{\partial f_1}{\partial Z_C}=0,\ \frac{\partial f_1}{\partial \psi}=0,$$
$$\frac{\partial f_1}{\partial \theta}=-l\sin\theta\cos\varphi,\ \frac{\partial f_1}{\partial \varphi}=-l\cos\theta\sin\varphi, \tag{11}$$

$$\frac{\partial f_2}{\partial X_C}=0,\ \frac{\partial f_2}{\partial Y_C}=1,\ \frac{\partial f_2}{\partial Z_C}=0,$$
$$\frac{\partial f_2}{\partial \psi}=l(\cos\psi\sin\theta\cos\varphi-\sin\psi\sin\varphi),$$
$$\frac{\partial f_2}{\partial \theta}=l\sin\psi\cos\theta\cos\varphi, \tag{12}$$
$$\frac{\partial f_2}{\partial \varphi}=l(-\sin\psi\sin\theta\sin\varphi+\cos\psi\cos\varphi),$$

the matrix of constraints reading

$$[\mathbf{B}]=$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & -ls\theta c\varphi & -lc\theta s\varphi \\ 0 & 1 & 0 & \begin{matrix} l(c\psi s\theta c\varphi \\ -s\psi s\varphi) \end{matrix} & \begin{matrix} ls\psi \\ c\theta c\varphi \end{matrix} & \begin{matrix} l(-s\psi s\theta s\varphi \\ +c\psi c\varphi) \end{matrix} \end{bmatrix}. \tag{13}$$

The coordinates of the point A are $x_A=-l$, $y_A=0$, $z_A=0$ (relative to the mobile reference frame $Cxyz$) and it results

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix}=\begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix}+[\mathbf{A}]\begin{bmatrix} -l \\ 0 \\ 0 \end{bmatrix}, \tag{14}$$

wherefrom

$$\begin{aligned} X_A &= X_C-l\cos\psi\cos\varphi, \\ Y_A &= Y_C-l(\sin\psi\sin\theta\sin\varphi+\cos\psi\sin\varphi), \\ Z_A &= Z_C-l(\cos\psi\sin\theta\cos\varphi+\sin\psi\sin\varphi). \end{aligned} \tag{15}$$

One may write

$$AP_1=\sqrt{(X_{P_1}-X_A)^2+(Y_{P_1}-Y_A)^2+(Z_{P_1}-Z_A)^2},$$
$$CP_2=\sqrt{(X_{P_2}-X_C)^2+(Y_{P_2}-Y_C)^2+(Z_{P_2}-Z_C)^2} \tag{16}$$
$$,$$
$$AP_3=\sqrt{(X_{P_3}-X_A)^2+(Y_{P_3}-Y_A)^2+(Z_{P_3}-Z_A)^2},$$

$$[\mathbf{Q}]=\begin{bmatrix} \cos\varphi\cos\theta & \sin\varphi & 0 \\ -\sin\varphi\cos\theta & \cos\varphi & 0 \\ \sin\theta & 0 & 1 \end{bmatrix}, \tag{17}$$

$$[\mathbf{r}_A]=\begin{bmatrix} 0 & -z_A & y_A \\ z_A & 0 & -x_A \\ -y_A & x_A & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & l \\ 0 & -l & 0 \end{bmatrix},$$
$$[\mathbf{r}_C]=\begin{bmatrix} 0 & -z_C & y_C \\ z_C & 0 & -x_C \\ -y_C & x_C & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}}=l$$
$$\begin{bmatrix} \begin{matrix} s\theta c\theta \\ (s\varphi-s\psi) \end{matrix} & \begin{matrix} -s\theta(-s\psi s\theta s\varphi \\ +c\psi c\varphi) \\ +s^2\psi c^2\theta \end{matrix} & \begin{matrix} -s\theta(c\psi s\theta s\varphi \\ +s\psi c\varphi) \\ -s\psi c\psi c^2\theta \end{matrix} \\ s\theta c\varphi & -s\psi c\theta c\varphi & c\psi c\theta c\varphi \\ c\theta s\varphi & \begin{matrix} s\psi s\theta s\varphi \\ -c\psi c\varphi \end{matrix} & \begin{matrix} -c\psi s\theta s\varphi \\ -s\psi c\varphi \end{matrix} \end{bmatrix}, \tag{19}$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_C][\mathbf{A}]^{\mathrm{T}}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{20}$$

$$\{\mathbf{F}\}_1=\frac{k(AP_1-l)}{AP_1}\begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} X_{P_1}-X_A \\ Y_{P_1}-Y_A \\ Z_{P_1}-Z_A \end{bmatrix}, \tag{21}$$

$$\{\mathbf{F}\}_2=\frac{2k(CP_2-l)}{CP_2}\begin{bmatrix} [\mathbf{I}] \\ [\mathbf{0}] \end{bmatrix}\begin{bmatrix} X_{P_2}-X_C \\ Y_{P_2}-Y_C \\ Z_{P_2}-Z_C \end{bmatrix}, \tag{22}$$

$$\{\mathbf{F}\}_3=\frac{k(AP_3-2l)}{AP_3}\begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{r}_A][\mathbf{A}]^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} X_{P_3}-X_A \\ Y_{P_3}-Y_A \\ Z_{P_3}-Z_A \end{bmatrix}, \tag{23}$$

$$\{\mathbf{F}\}_4=\begin{bmatrix} 0 & 0 & -mg & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{24}$$

$$\{\mathbf{F_q}\}=\{\mathbf{F}\}_1+\{\mathbf{F}\}_2+\{\mathbf{F}\}_3+\{\mathbf{F}\}_4`, \tag{25}$$

$$[\mathbf{J}]=\begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}, \tag{26}$$

$$J_y=J_z=\frac{ml^2}{3}, \tag{27}$$

$$[\mathbf{m}]=\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \tag{28}$$

$$[\mathbf{S}]=[\mathbf{0}], \tag{29}$$

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}]=$$
$$\begin{bmatrix} J_x c^2\varphi + J_y\left(s^2\varphi c^2\theta+s^2\theta\right) & \left(J_x-J_y\right) s\varphi c\varphi c\theta & J_y s\theta \\ \left(J_x-J_y\right) s\varphi c\varphi c\theta & J_x s^2\varphi + J_y c^2\varphi & 0 \\ J_y s\theta & 0 & J_y \end{bmatrix}, \tag{30}$$

$$[\mathbf{M}]=\begin{bmatrix} [\mathbf{m}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\mathbf{Q}] \end{bmatrix}, \tag{31}$$

$$\{\boldsymbol{\omega}\}=\begin{bmatrix} \dot\psi\cos\varphi\cos\theta+\dot\theta\sin\varphi \\ -\dot\psi\sin\varphi\cos\theta+\dot\theta\cos\varphi \\ \dot\psi\sin\theta+\dot\varphi \end{bmatrix}=\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \tag{32}$$

$$[\boldsymbol{\omega}]=\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \tag{33}$$

$$\{\widetilde{\mathbf{F}}_{\mathbf{s}}\}=[0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{34}$$

$$[\mathbf{Q}_\varphi]=\begin{bmatrix} -\sin\varphi\cos\theta & \cos\varphi & 0 \\ -\cos\varphi\cos\theta & -\sin\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{35}$$

$$[\mathbf{Q}_\theta]=\begin{bmatrix} -\cos\varphi\sin\theta & 0 & 0 \\ \sin\varphi\sin\theta & 0 & 0 \\ \cos\theta & 0 & 0 \end{bmatrix}, \tag{36}$$

$$[\dot{\mathbf{Q}}]=\dot\varphi[\mathbf{Q}_\varphi]+\dot\theta[\mathbf{Q}_\theta], \tag{37}$$

$$\{\widetilde{\mathbf{F}}_{\boldsymbol{\beta}}\}=-\left[[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_C][\dot{\mathbf{Q}}]+[\mathbf{Q}][\boldsymbol{\omega}][\mathbf{J}_C][\mathbf{Q}]\right]\{\dot{\boldsymbol{\beta}}\}, \tag{38}$$

$$\{\widetilde{\mathbf{F}}_{\mathbf{q}}\}=\left[\{\widetilde{\mathbf{F}}_{\mathbf{s}}\}^{\mathrm{T}} \quad \{\widetilde{\mathbf{F}}_{\boldsymbol{\beta}}\}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{39}$$

$$\{\boldsymbol{\lambda}\}=[\lambda_1 \quad \lambda_2]^{\mathrm{T}}, \tag{40}$$

$$[\dot{\mathbf{B}}]=\begin{bmatrix} 0 & 0 & 0 & 0 & \dot b_{15} & \dot b_{16} \\ 0 & 0 & 0 & \dot b_{24} & \dot b_{25} & \dot b_{26} \end{bmatrix}, \tag{41}$$

$$\dot b_{15}=-l\dot\theta\cos\theta\cos\varphi+l\dot\varphi\sin\theta\sin\varphi, \tag{42}$$

$$\dot b_{16}=l\dot\theta\sin\theta\sin\varphi-l\dot\varphi\cos\theta\cos\varphi, \tag{43}$$

$$\begin{aligned}\dot b_{24}=l(&-\dot\psi\sin\psi\sin\theta\cos\varphi \\ &+\dot\theta\cos\psi\cos\theta\cos\varphi \\ &-\dot\varphi\cos\psi\sin\theta\sin\varphi \\ &-\dot\psi\cos\psi\sin\varphi-\dot\varphi\sin\psi\cos\varphi),\end{aligned} \tag{44}$$

$$\begin{aligned}\dot b_{25}=l(&\dot\psi\cos\psi\cos\theta\cos\varphi \\ &-\dot\theta\sin\psi\sin\theta\cos\varphi \\ &-\dot\varphi\sin\psi\cos\theta\cos\varphi),\end{aligned} \tag{45}$$

$$\begin{aligned}\dot b_{26}=l(&-\dot\psi\cos\psi\sin\theta\cos\varphi \\ &-\dot\theta\sin\psi\cos\theta\sin\varphi \\ &-\dot\varphi\sin\psi\sin\theta\cos\varphi \\ &-\dot\psi\sin\psi\cos\varphi-\dot\varphi\cos\psi\sin\varphi),\end{aligned} \tag{46}$$

$$\begin{aligned}\{\mathbf{q}\}&=[X_O \quad Y_O \quad Z_O \quad \psi \quad \theta \quad \varphi]^T, \\ \{\dot{\mathbf{q}}\}&=[\dot X_O \quad \dot Y_O \quad \dot Z_O \quad \dot\psi \quad \dot\theta \quad \dot\varphi]^T.\end{aligned} \tag{47}$$

The matrix differential equation of motion reads

$$\begin{bmatrix} [\mathbf{M}] & -[\mathbf{B}]^T \\ [\mathbf{B}] & [\mathbf{0}] \end{bmatrix}\begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \{\boldsymbol{\lambda}\} \end{bmatrix}=\begin{bmatrix} \{\mathbf{F}_{\mathbf{q}}\}+\{\widetilde{\mathbf{F}}_{\mathbf{q}}\} \\ -[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}. \tag{48}$$

For the equilibrium positions one has to solve the equation

$$\{\mathbf{F}_{\mathbf{q}}\}+[\mathbf{B}]^T\{\boldsymbol{\lambda}\}=\{\mathbf{0}\}, \tag{49}$$

that is

$$\begin{bmatrix} F_{q_1} \\ F_{q_2} \\ F_{q_3} \\ F_{q_4} \\ F_{q_5} \\ F_{q_6} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \\ b_{14} & b_{24} \\ b_{15} & b_{25} \\ b_{16} & b_{26} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{50}$$

At this equation one adds the rotations

$$f_1(X_C, Y_C, Z_C, \psi, \theta, \varphi)=X_C+l\cos\theta\cos\varphi=0, \tag{51}$$

$$f_2\left(X_C, Y_C, Z_C, \psi, \theta, \varphi\right) = Y_C + l\left(\sin\psi\sin\theta\cos\varphi + \cos\psi\sin\varphi - 2\right) = 0\,, \tag{52}$$

obtaining a system of eight non-linear equations with axis Cx is an useless degree of freedom. One may choose one of the following selections:

i) one eliminates the parameter φ from all matrices erasing the corresponding line and column;

ii) one considers that $J_x \neq 0$ and the initial conditions $\varphi_0 = 0$, $\dot{\varphi}_0 = 0$ resulting $\varphi(t) = 0$ for any time t.

3. EXAMPLE 2

The homogenous quadratic shell $ABCD$ of dimension $AB = 2l$ is situated in a vertical plane and is connected to the spring AP_1 and DP_2, where $P_1(0,3l)$, $P_2(3l,4l)$. The point O (center of weight) is constrained to move along the straight line (d) of equation

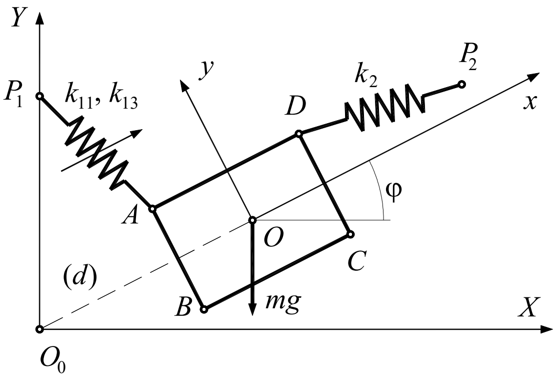

Figure 2. Example 2.

$$Y = X\,. \tag{53}$$

The mass of the shell is m and the non-deformed lengths of the springs are l and $2l$.

The first spring is a non-linear one, the elastic force being

$$F_1 = k_{11}\Delta l_1 + k_{13}\left(\Delta l_1\right)^3, \tag{54}$$

while the second spring is a linear one, the elastic force writing

$$F_1 = k_2 \Delta l_2\,. \tag{55}$$

In this situation, one gets for the first spring

$$\{\mathbf{F}\}_1 = \left\{ \frac{k_{11}\left(AP_1 - l\right)}{AP_1} \begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^T [\mathbf{r}_A][\mathbf{A}]^T \end{bmatrix} + \frac{k_{13}\left(AP_1 - l\right)^3}{DP_1} \begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^T [\mathbf{r}_A][\mathbf{A}]^T \end{bmatrix} \right\} \begin{bmatrix} X_{P_1} - X_A \\ Y_{P_1} - Y_A \\ Z_{P_1} - Z_A \end{bmatrix}, \tag{56}$$

while for the second spring one obtains

$$\{\mathbf{F}\}_2 = \frac{k_2\left(DP_2 - 2l\right)}{DP_2} \begin{bmatrix} [\mathbf{I}] \\ [\mathbf{Q}]^T [\mathbf{r}_A][\mathbf{A}]^T \end{bmatrix} \begin{bmatrix} X_{P_2} - X_D \\ Y_{P_2} - Y_D \\ Z_{P_2} - Z_D \end{bmatrix}. \tag{57}$$

We have

$$\{\mathbf{F}\}_3 = \begin{bmatrix} 0 & -mg & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{58}$$

$$\{\mathbf{F_q}\} = \{\mathbf{F}_1\} + \{\mathbf{F}_2\} + \{\mathbf{F}_3\}, \tag{59}$$

$$[\mathbf{A}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{60}$$

$$[\mathbf{S}] = [\mathbf{0}], \tag{61}$$

$$\{\boldsymbol{\omega}\} = \begin{bmatrix} 0 & 0 & \dot{\varphi} \end{bmatrix}^T, \tag{62}$$

$$\{\boldsymbol{\omega}\} = \begin{bmatrix} 0 & -\dot{\varphi} & 0 \\ \dot{\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \tag{63}$$

$$\{\widetilde{\mathbf{F}}_\mathbf{s}\} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \tag{64}$$

$$\{\mathbf{J}_O\} = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}, \; J_z = \frac{2ml^3}{3}, \tag{65}$$

$$\{\widetilde{\mathbf{F}}_{\boldsymbol{\beta}}\} = -\left[[Q]^T [\mathbf{J}_O][\dot{\mathbf{Q}}] + [\mathbf{Q}]^T + [\boldsymbol{\omega}][\mathbf{J}_O][\mathbf{Q}]\right] \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix}, \tag{66}$$

$$\{\widetilde{\mathbf{F}}_\mathbf{q}\} = -\begin{bmatrix} \{\widetilde{\mathbf{F}}_\mathbf{q}\}^T & \{\widetilde{\mathbf{F}}_{\boldsymbol{\beta}}\}^T \end{bmatrix}^T. \tag{67}$$

Since point O belongs to the straight line (d) one gets the constrain function

$$f(X_O, Y_O, \varphi) = Y_O - X_O = 0, \tag{68}$$

wherefrom

$$\frac{\partial f}{\partial X_O} = -1, \; \frac{\partial f}{\partial Y_O} = 1, \; \frac{\partial f}{\partial \varphi} = 0. \tag{69}$$

The matrix of constrain reads

$$[\mathbf{B}] = [-1 \quad 1 \quad 0], \tag{70}$$

hence

$$[\dot{\mathbf{B}}] = [0 \quad 0 \quad 0]. \tag{71}$$

We denote by $\{\mathbf{F}^*\}$ the matrix obtained from $\{\mathbf{F_q}\} + \{\tilde{\mathbf{F}}_\mathbf{q}\}$ retaining only the first, the second and the sixth line.

It results the matrix differential equation of motion

$$\begin{bmatrix} m & 0 & 0 & 1 \\ 0 & m & 0 & -1 \\ 0 & 0 & J_O & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{X}_O \\ \ddot{Y}_O \\ \ddot{\varphi} \\ \lambda \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}^*\} \\ 0 \end{bmatrix}. \tag{72}$$

One may choose to use all the equations considering that the system is a spatial one. In this case one has to use the following constrain functions

$$f_1(X_O, Y_O, Z_O, \psi, \theta, \varphi) = Y_O - X_O = 0, \tag{73}$$

$$f_2(X_O, Y_O, Z_O, \psi, \theta, \varphi) = Z_O = 0, \tag{74}$$

$$f_3(X_O, Y_O, Z_O, \psi, \theta, \varphi) = Z_A = 0, \tag{75}$$

$$f_4(X_O, Y_O, Z_O, \psi, \theta, \varphi) = Z_D = 0. \tag{76}$$

The constrain functions f_3 and f_4 are equivalent to the following ones

$$f_3^*(X_O, Y_O, Z_O, \psi, \theta, \varphi) = \psi = 0, \tag{77}$$

$$f_4^*(X_O, Y_O, Z_O, \psi, \theta, \varphi) = \theta = 0 \tag{78}$$

Using the last functions, one obtains the matrix of constraints

$$[\mathbf{B}^*] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{79}$$

4. NUMERICAL RESULTS

For the first example we consider the following values: $l = 0.2\,\mathrm{m}$, $m = 1.0\,\mathrm{kg}$, $J_x = 1.0\,\mathrm{kgm}^2$, $J_y = \frac{ml^2}{3}$, $J_z = \frac{ml^2}{3}$, $k = 50.0\,\mathrm{N/m}$. The simulation is performed for a period of time equal to $t_{\max} = 1.0\,\mathrm{s}$, the step of integration being $\Delta t = 10^{-3}\,\mathrm{s}$. The initial values are: $X_O^0 = 0\,\mathrm{m}$, $Y_O^0 = 0.2\,\mathrm{m}$, $Z_O^0 = 0.2\,\mathrm{m}$, $\psi^0 = 0\,\mathrm{rad}$, $\theta^0 = 0\,\mathrm{rad}$, $\varphi^0 = \frac{\pi}{2}\,\mathrm{rad}$, $\dot{X}_O^0 = 0\,\mathrm{m/s}$, $\dot{Y}_O^0 = 0\,\mathrm{m/s}$, $\dot{Z}_O^0 = 0\,\mathrm{m/s}$, $\dot{\psi}^0 = 0\,\mathrm{rad/s}$, $\dot{\theta}^0 = 0\,\mathrm{rad/s}$, $\dot{\varphi} = 0\,\mathrm{rad/s}$.

The results of the simulation are captured in the next diagrams.

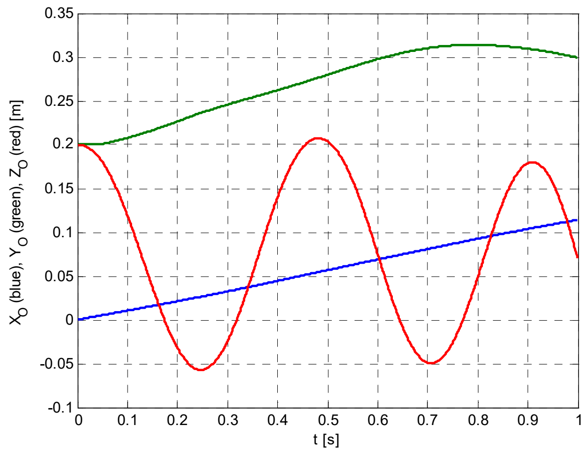

Figure 3. Time histories $X_O = X_O(t)$, $Y_O = Y_O(t)$ and $Z_O = Z_O(t)$.

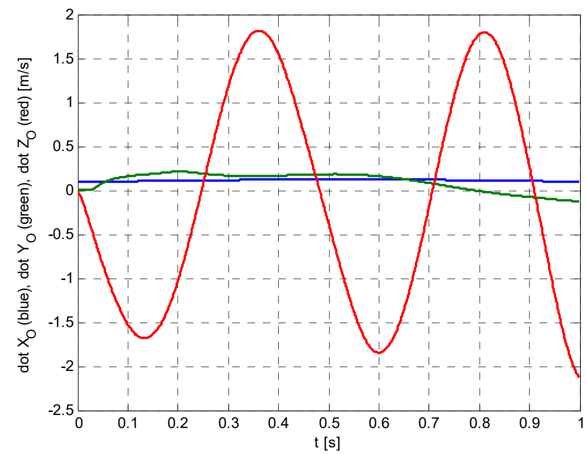

Figure 4. Time histories for $\dot{X}_O = \dot{X}_O(t)$, $\dot{Y}_O = \dot{Y}_O(t)$ and $\dot{Z}_O = \dot{Z}_O(t)$

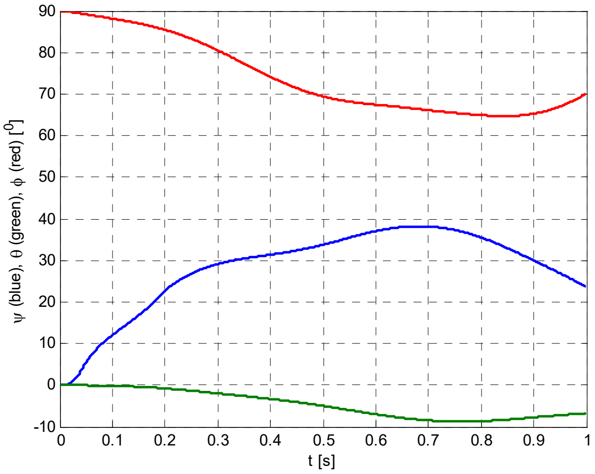

Figure 5. Time histories $\psi=\psi(t)$, $\theta=\theta(t)$ and $\varphi=\varphi(t)$.

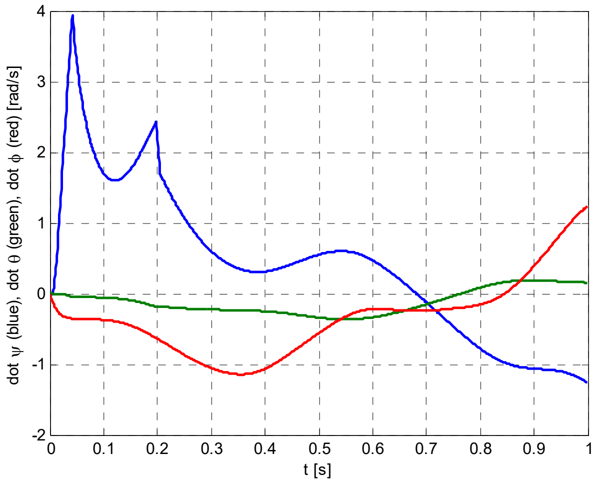

Figure 6. Time histories $\dot\psi=\dot\psi(t)$, $\dot\theta=\dot\theta(t)$ and $\dot\varphi=\dot\varphi(t)$.

For the second example we will discuss the equilibrium positions. We reformulate the problem as follows: Is it possible a particular solution? Which are conditions to be fulfilled by the input parameters in order to obtain that equilibrium position?

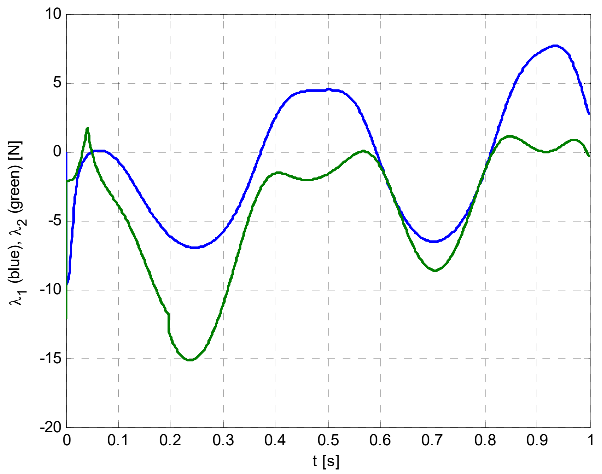

Figure 7. Time histories $\lambda_1=\lambda_1(t)$ and $\lambda_2=\lambda_2(t)$.

We will start with the position characterized by $X_O=l$, $Y_O=l$, $\varphi=0$. In this position we get $X_A=0$, $Y_A=2l$, $X_D=2l$, $Y_D=2l$, $AP_1=l$, $DP_2=l\sqrt{5}$,

$$\{\mathbf{F}_1\}=[0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{80}$$

$$\{\mathbf{F}_2\}=\frac{k_2\left(\sqrt{5}-2\right)}{\sqrt{5}}\begin{bmatrix} l \\ 2l \\ 3l^2 \end{bmatrix}, \tag{81}$$

$$\{\mathbf{F}_3\}=[0 \quad -mg \quad 0]^{\mathrm{T}}, \tag{82}$$

$$\{\mathbf{N}\}=[\lambda \quad -\lambda \quad 0]^{\mathrm{T}}, \tag{83}$$

where $\mathbf{N}$ is the normal reaction at the point O.

From the equation of equilibrium

$$\{\mathbf{F}_1\}+\{\mathbf{F}_2\}+\{\mathbf{F}_3\}+\{\mathbf{N}\}=[0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{84}$$

retaining only the third line, one obtains

$$\frac{k_2\left(\sqrt{5}-1\right)}{\sqrt{5}}3l^2=0, \tag{85}$$

wherefrom $k_2=0$, that is an impossible value. It results that this equilibrium position can not be achieved.

Another case which will be considered in this paper is that one characterized by $X_O=l$, $Y_O=l$, $\varphi=\frac{\pi}{4}\,\mathrm{rad}$. In this situation one gets $X_A=l\left(1-\sqrt{2}\right)$, $Y_A=l$, $X_D=l$, $Y_D=l\left(1+\sqrt{2}\right)$, $AP_1=l\sqrt{7-\sqrt{2}}$, $DP_2=l\sqrt{15-6\sqrt{2}}$,

$$\{\mathbf{F}_1\}=$$
$$\left[\frac{k_{11}\left(AP_1-l\right)}{AP_1}+\frac{k_{13}\left(AP_1-l\right)^3}{AP_1}\right]\begin{bmatrix} l\left(\sqrt{2}-1\right) \\ 2l \\ l^2\left(2+\sqrt{2}\right) \end{bmatrix}, \tag{86}$$

$$\{\mathbf{F}_2\}=\frac{k_2\left(DP_2-l\right)}{DP_2}\begin{bmatrix} 2l \\ l\left(3-\sqrt{2}\right) \\ l^2\left(3\sqrt{2}-2\right) \end{bmatrix}, \tag{87}$$

$$\{\mathbf{F}_3\}=[0 \quad -mg \quad 0]^{\mathrm{T}}, \tag{88}$$

$$\{\mathbf{N}\} = [\lambda \quad -\lambda \quad 0]^{\mathrm{T}} . \tag{89}$$

Denoting

$$A = \frac{k_{11}(AP_1 - l)}{AP_1} + \frac{k_{13}(AP_1 - l)^3}{AP_1}, \tag{90}$$

$$B = \frac{k_2(DP_2 - l)}{DP_2}, \tag{91}$$

one gets the system

$$\begin{gathered}(\sqrt{2}-1)Al + 2Bl - \lambda = 0, \\ 2Al + (3-\sqrt{2})Bl - mg + \lambda = 0, \\ (2+\sqrt{2})Al^2 + (3\sqrt{2}-2)Bl^2 = 0 .\end{gathered} \tag{92}$$

This system leads to

$$\begin{gathered}B = \frac{mg}{2l\sqrt{2}} > 0, \\ A = \frac{B(2-3\sqrt{2})}{2+\sqrt{2}} = \frac{mg(2-3\sqrt{2})}{4l(1+\sqrt{2})} < 0, \\ \lambda = Al(\sqrt{2}-1) + 2Bl = \\ \frac{mg(2-3\sqrt{2})(\sqrt{2}-1)}{4(1+\sqrt{2})} + \frac{mg}{\sqrt{2}} .\end{gathered} \tag{93}$$

Taking into account the previous expressions, one gets the relations

$$k_2 = \frac{mg\sqrt{15-6\sqrt{2}}}{2l\sqrt{2}\left(\sqrt{15-6\sqrt{2}}-1\right)}, \tag{94}$$

$$\begin{gathered}\frac{k_{11}l\left(\sqrt{7-\sqrt{2}}-1\right)}{\sqrt{7-\sqrt{2}}} + \frac{k_{13}l^3\left(\sqrt{7-\sqrt{2}}-1\right)^3}{\sqrt{7-\sqrt{2}}} = \\ mg(2-3\sqrt{2}),\end{gathered} \tag{95}$$

that is

$$k_2 \approx 0.5813\frac{mg}{l}, \tag{96}$$

$$0.5769k_{11}l + 1.0724k_{13}l^3 \approx -2.2426mg . \tag{97}$$

This equilibrium position is possible if the relations (96) and (97) hold true. The condition $k_{13} < 0$ implies that first spring has a soft characteristic.

For instance, assuming $m = 3.0\,\mathrm{kg}$, $l = 0.15\,\mathrm{m}$, the gravitational acceleration $g = 9.8065\,\mathrm{m/s^2}$, and $k_{11} = k_2$, it results the values $k_2 \approx 114.0104\,\mathrm{N/m}$, $k_{11} = k_2 \approx 114.0104\,\mathrm{N/m}$, $k_{13} \approx -70.7218\,\mathrm{N/m^3}$, $\lambda \approx 17.9727\,\mathrm{N}$, while the reactions at the point O are $N_X = -\lambda \approx -17.9727\,\mathrm{N}$, $N_Y = \lambda \approx 17.9727\,\mathrm{N}$, $N = \lambda\sqrt{2} \approx 25.4173\,\mathrm{N}$.

5. CONCLUSIONS

In this paper we presented two examples, the first a spatial one and the second a planar one, in order to highlight the theory described in our previous paper. A great attention is paid to the obtaining of the matrix of constraints.

Because of the errors of calculation, the simulation has to be limited to small values of the variable time. Some quasi-periodic evolutions may be observed for certain variables.

The initial conditions have to be considered according to the constraint functions. In the general case this is a huge problem.

The second example shows that there is the possibility of existence of some positions which verify the constraints but they cannot be equilibrium positions. It is the case of the first situation in Example 2. The second case highlights the situation in which a equilibrium is possible only for particular values of the input parameters.

Taking into account the non-linearity of the system used for the determination of the equilibrium positions and the results presented above, one may state that in the general case the system may have zero, one or several equilibrium positions.

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