Designing Quasi-harmonic Circular Membranes with Different Densities on Angular Sectors

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Abstract: - We develop a mathematical method to design new circular drum membranes vibrating with minimal disharmony. The membranes considered have different areal mass density, first over two sectors of complementary central angles, and second over *N* sectors of the same central angle. We exemplify the method with membranes having two or three different densities.

Keywords: - optimal design, circular membranes, angular sectors, musical instruments

1. INTRODUCTION

Unlike stringed instruments, circular drums with a uniform density membrane produce disharmonic sounds as a rule, because their natural frequencies are not integer multiples of one of these. However, it is possible to make these sounds quasi-harmonic by modifying the membrane mass density. An example of this is the Indian tabla whose membrane has a central disc with a radius equal to 40% that of the entire membrane and where the mass density is about ten times that of the rest of the membrane [1, 2]. Since this reduction of the disharmony of the membrane sounds results from a modification of its density over a region delimited by a curve belonging to one of the two sets of nodal lines of circular membranes, that of concentric circles, it is natural to ask what can be obtained by modifying the density over one or many regions delimited by curves belonging to the other set of its nodal lines, namely its diameters.

This work aims to determine theoretically the physical parameters of circular membranes whose vibrations would produce sounds with minimal disharmony. We shall develop a mathematical model of a circular membrane with different uniform densities, first over two sectors of complementary central angles, and second over a set of sectors having the same central angle. From these models, we shall derive conditions allowing us to determine the configurations of these membranes that minimize the disharmony of their vibrations.

The quality of sounds produced by a circular membrane strongly depends on the degree of harmony of its vibrations, but other properties contribute to their musical qualities. One of these is the timbre, which is determined by the number of different natural frequencies of the vibrations involved to produce a sound, as well as their respective amplitudes. The timbre of a sound is said to be high if it consists of many vibrations with different natural frequencies, and low otherwise. Our evaluation of the musical qualities of configurations of circular membranes with one or several distinct angular sectors will consider both degree of harmony and timbre of their vibrations.

The rest of this paper is as follows. In Section 2, we construct a mathematical model to study the degree of harmony of a circular membrane having one angular sector whose density is different from that elsewhere on the membrane. Using this model, we determine the configurations allowing the membrane to vibrate with a minimum of disharmony. In Section 3, we extend the method of Section 2 to minimize the disharmony of a membrane made up of *N* different density angular sectors of 360/*N* degrees. The results of the numerical optimization methods of Sections 2 and 3 are presented as tables gathering results with common features. Finally, Section 4 presents the best configurations of theoretically easy to build membranes with angular sectors of different densities that are likely to show musical qualities beyond those of uniform density membranes.

2. MEMBRANES WITH ONE DISTINCT ANGULAR SECTOR

2.1. Mathematical model

Let D be a circular membrane of unit radius formed by an angular sector D_0, $0 \leq \theta \leq \theta^*$, of density ρ_0, and its complementary angular sector D_1, $\theta^* \leq \theta \leq 2\pi$, of density ρ_1 (see Fig.1). Let $\psi(r,\theta,t)$ be a twice-differentiable function describing the small displacements of the point of polar coordinates (r,θ) on D at time t, with respect to its rest position. The domain of definition of this function is $[0,1] \times \boldsymbol{R} \times \boldsymbol{R}_+$. Also, let $\psi_k(r,\theta,t)$ be twice-differentiable functions with the same domain as $\psi(r,\theta,t)$ and which coincide with the latter on D_k for $k = 0,1$, respectively. Using Hamilton's principle, it is straightforward to show that the functions $\psi_k(r,\theta,t)$ must satisfy:

$$\partial_t^2 \psi_k = c_k^2\left(\partial_r^2 \psi_k + \frac{1}{r}\partial_r \psi_k + \frac{1}{r^2}\partial_\theta^2 \psi_k\right),\ k = 1, 2, \quad (1)$$

where $c_k^2 = \tau_k / \rho_k$ and $\tau_k = T_k / h_k$ in which T_k is the radial tension applied to the circular contour and h_k is the thickness of the sector k of the membrane [3]. We also have the equations expressing what happens at the junctions between the sectors

$$\psi_0(r,0,t) = \psi_1(r,2\pi,t), \quad (2)$$

$$\psi_0(r,\theta^*,t) = \psi_1(r,\theta^*,t), \quad (3)$$

and

$$\partial_\theta \psi_0(r,0,t) = \partial_\theta \psi_1(r,2\pi,t),\ r \neq 0, \quad (4)$$

$$\partial_\theta \psi_0(r,\theta^*,t) = \partial_\theta \psi_1(r,\theta^*,t),\ r \neq 0, \quad (5)$$

as well as the boundary conditions

$$\psi_0(1,\theta,t) = 0, 0 \leq \theta \leq \theta^*, \quad (6)$$

$$\psi_1(1,\theta,t) = 0, \theta^* \leq \theta \leq 2\pi, \quad (7)$$

to which usually add initial conditions, but the latter will not be necessary to determine the physical parameters minimizing the disharmony of membrane vibrations.

Since the membrane is circular, the functions ψ_k can be written as

$$\psi_k(r,\theta,t) = R_k(r)\Theta_k(\theta)T_k(t),\ k = 0,1, \quad (8)$$

Substituting (8) into (1), we easily find that

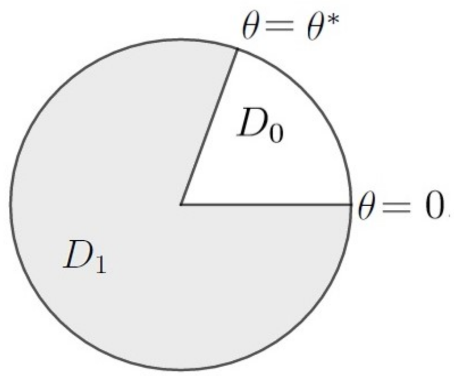

Figure 1. Schematic of a circular membrane with one distinct angular sector.

$$T_k''(t) + \lambda_k^2 c_k^2 T_k(t) = 0, \quad (9)$$

$$\Theta''(\theta) + \mu_k^2 \Theta(\theta) = 0, \quad (10)$$

and

$$r^2 R_k''(r) + rR_k'(r) + (\lambda_k^2 r^2 - \mu_k^2)R_k(r) = 0, \quad (11)$$

where λ_k and μ_k are separation constants. The solutions of (9), (10) and (11) can be written respectively as

$$T_k(t) = A_k \sin(\lambda_k c_k t + \alpha_k), \quad (12)$$

$$\Theta_k(\theta) = B_k \sin(\mu_k\theta + \beta_k), \quad (13)$$

and

$$R_k(r) = C_k J_{\mu_k}(\lambda_k r), \quad (14)$$

where $-\pi/\lambda_k c_k \leq \alpha_k \leq \pi/\lambda_k c_k$ are the phases related to the temporal sinusoidal movement for $\lambda_k \neq 0$, $-\pi/\mu_k \leq \alpha_k \leq \pi/\mu_k$ are the phases related to the spatial sinusoidal movement for $\mu_k \neq 0$, and J_{μ_k} are the Bessel functions of the first kind and of order μ_k. Also, A_k, B_k and C_k denote arbitrary constants.

Since we want to determine the physical parameters of a circular membrane whose vibrations produce minimal disharmony, it is natural to argue that these vibrations are spatially 2π-periodic, i.e. that the functions Θ_k satisfy

$$\Theta_k(\theta + 2\pi) = \Theta_k(\theta)., \quad (15)$$

By applying (13) to (15), we easily get $\mu_k = m_k$, where $m_k \in \boldsymbol{Z}$. Since μ_k stands for the order of J_{μ_k}, we can set $m_k \in \boldsymbol{N}$. According to (13), the value of m_k is the number of nodal diameters formed by the membrane vibrations.

Substituting (8) into (2), we obtain

$$\frac{T_0(t)}{T_1(t)} = \frac{\Theta_1(2\pi)}{\Theta_0(0)} \frac{R_1(r)}{R_0(r)}, \tag{16}$$

Since the left-hand side of (16) is a function of t while its right-hand side is a function of r, each of these must be equal to a same constant K. Therefore,

$$T_0(t) = K\, T_1(t), \tag{17}$$

and

$$R_1(r) = \frac{K\, \Theta_0(0)}{\Theta_1(2\pi)} R_0(r), \tag{18}$$

Unlike to what happens when applying the usual method of separation of variables, which involves derivatives of single-variable functions and where the separation constant leads to the eigenvalues of the problem under consideration, here the constant K only shows the links between T_0 and T_1 on the one hand and between R_0 and R_1 on the other hand. This constant disappears when is expressed the link between ψ_0 and ψ_1 resulting from these. From (17), we get that sectors 1 and 2 vibrate in unison, that is with the same temporal phase and period. Using (12), we then get $\alpha_0 = \alpha_1$ and

$$\lambda_0 c_0 = \lambda_1 c_1, \tag{19}$$

Substituting (14) into (18), and using the independence of J_{m_0} from J_{m_1} when $m_0 \neq m_1$, it follows that $m_0 = m_1 = m \in \boldsymbol{N}$.

By applying the preceding results to (2)-(5), it is straightforward to show that $\Theta_0 = \Theta_1$ for all $m \in \boldsymbol{N}$. But for any $\theta \in [0, \theta^*]$, we have $\Theta_0(\theta + \theta^*) = \Theta_1(\theta)$. This leads to

$$\Theta_0(\theta + \theta^*) = \Theta_0(\theta), \tag{20}$$

for any $\theta^* \in [0, 2\pi[$. The substitution of (13) into (20) finally shows that m and θ^* must satisfy

$$m\, \theta^* = 2\pi\, n, \tag{21}$$

for $n \in \boldsymbol{N}$. Eq. (21) is the eigenvalue equation of the present boundary-value problem.

To take $\theta^* = 0$ corresponds to set $n = 0$ into (21). In this case, the membrane has only one density and there is no restriction on m. We therefore assume that $0 < \theta^* < 2\pi$. If $n = 0$, then $m = 0$ in (21). However, for $n > 0$ we have $m > 0$ and thus $0 < n/m < 1$, because $0 < \theta^*/2\pi < 1$. Let m_*, n_* be the smallest positive integers such that $n_*/m_* = n/m$. These integers are then coprime. The vibrations of the membrane will thus be periodic in space if $m = lm_*$ for $l \in \boldsymbol{N}$.

We now examine the consequences of (19) on the periodicity in time of the membrane vibrations. Since $m_0 = m_1 = m \in \boldsymbol{N}$, then the zeros λ_k of $J_{\mu_k}(x)$ in (12) and (14) are independent of k and therefore can be denoted λ. Assuming that the values of τ_k are the same on both sides of the radii delimiting the angular sector, Eq. (19) can be written as

$$\frac{\lambda^{(0)}}{\lambda^{(1)}} - \sigma = 0, \tag{22}$$

where $\lambda^{(0)}$ and $\lambda^{(1)}$ are two zeros of $J_{\mu_k}(x)$ and $\sigma = \sqrt{\rho_0/\rho_1}$.

Therefore, a circular membrane having an angular sector $\theta^* \in [0, 2\pi[$ with a distinct density will have spatial periodic vibrations if $\theta^* = 2\pi\, n_*/m_*$, where m_*, $n_* \in \boldsymbol{N}^*$, $n_* < m_*$ and m_*, n_* are coprime. The vibrations will then be described by the functions $J_m(x)$, where $m = lm_*$ for $l \in \boldsymbol{N}$. Moreover, these vibrations will also be periodic in time with angular frequency ω only if ρ_0 and ρ_1 are such that $\omega = \lambda^{(0)} c_0 = \lambda^{(1)} c_1$, where $\lambda^{(0)}$ and $\lambda^{(1)}$ are distinct zeros of $J_m(x)$. But for any choice of ρ_0 and ρ_1, it is impossible to satisfy (22) for all pairs of different zeros of $J_m(x)$. However, since high overtones are greatly damped due to the weakness of their energies, we shall restrict ourselves to the orders m of $J_m(x)$ whose zeros lead to frequencies near the fundamental or its first four harmonics. This number of harmonics is consistent with previous studies [4-6].

2.2. Numerical method and results

To facilitate the numerical solution of the above optimization problem, we limit θ^* to take an integer number of degrees between 1 and 360. Since the order of the functions J_{lm*} involved in this solution is independent of n_* for each value of σ in (22), we shall determine the possibilities for m_* by setting $n_* = 1$. From (21), it then follows that m_* must be one of the 24 integer divisors of 360, from which we eliminate the divisor 1, because it corresponds to a uniform density membrane. We are thus led to consider the following values for θ^*: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120 and 180.

For any given $m = lm_* = m_l$, let us suppose that among the first five zeros of $J_{m_l}(x)$ we have I pairs of distinct zeros $(\lambda^{(0)}_{m_l i}, \lambda^{(1)}_{m_l i})$, $i = 1, 2, \ldots, I$, such that $\left|\frac{\lambda^{(0)}_{m_l i}}{\lambda^{(1)}_{m_l i}} - \sigma\right| \approx 0$.

Numerically, for each $\sigma \in [0.1, 3]$ covered with a step size of 0.01, we shall seek amid these pairs those satisfying

$$\left|\frac{\lambda^{(0)}_{m_l i}}{\lambda^{(1)}_{m_l i}} - \sigma\right| < 0.01, \qquad (23)$$

The value of the upper bound of the inequality (23) is inversely proportional to the order of the Bessel functions involved in the solution of this problem; here we have chosen 0.01 to moderate this order. When we check if two zeros λ^* and λ^{**} of $J_{m_l}(x)$ fulfill (23), we must consider separately the two pairs (λ^*, λ^{**}) and (λ^{**}, λ^*), because it is possible that the first pair does not satisfy (23) while the second one does it. When the two pairs satisfy (23), we keep only one of these. The angular frequency assigned to the pair ($\lambda^{(0)}_{m_l i}$, $\lambda^{(1)}_{m_l i}$), is $\omega_{m_l i} = \lambda^{(0)}_{m_l i} c_0$ for i = 1, 2, …, I. Doing this for l = 0, 1, 2, 3, 4, 5, we find a set of angular frequencies $\{\omega_{m_l i}\}$.

Let Λ be the smallest zero $\lambda^{(0)}_{m_l i}$ of the Bessel function associated with a given m_* that satisfies (23) for a given value of σ. Then Λ determines the membrane fundamental angular frequency Ω through the relation $\Omega = \Lambda c_0$. Setting $l = 0$ in $m_l = l m_*$, we see that the function J_0 is related to each value of m_*. But the smallest zero of $J_0(x)$ does not necessarily satisfy (23) and therefore does not necessarily determine the fundamental frequency of all possible configurations of the membrane. Also note that when m_* and l are fixed, then the ratios of the zeros of $J_{lm*}(x)$ satisfying (23) over the largest one in this set give the radii of nodal circles of the membrane.

We shall use a method of least squares weighted by the inverse of the harmonic rank to estimate the global deviation of the identified zeros $\lambda^{(0)}_{m_l i}$ from the values corresponding to the first five harmonics. This deviation is defined by the function

$$E(\theta^*, \sigma) = \sum_{h=1}^{5} \left(\frac{\lambda^{(0)}_{m_l i}}{h\Lambda} - 1 \right)^2, \qquad (24)$$

where θ^* takes the 23 already specified angles between 1 and 180 degrees, and the summation is for all zeros $\lambda^{(0)}_{m_l i}$ related to the frequencies contributing to the harmonics of ranks 1 to 5. In the application of (24), we associate with $h \in \{1, 2, 3, 4, 5\}$ all $\lambda^{(0)}_{m_l i}$ such that $\lambda^{(0)}_{m_l i}/\Lambda$ is in the interval $[h - 1/2, h + 1/2]$. Our objective is to determine the values of θ^* and σ that minimize (24).

Table 1. Characteristics of membranes with one distinct angular sector showing the lowest values of E and one to five consecutive quasi-harmonics. Data in the last horizontal segment applies to tabla membranes [5].

θ^*	σ	Λ	ω_1	ω_2	ω_3	ω_4	ω_5	E
1-180	0.16 0.17	2.405	$1.000^{(180,1)}$					0.000
36 72 180	0.50 0.51 1.99 2.00	14.48	$1.000^{(5,1)}$	$1.996^{(5,2)}$				0.000
3	0.91	129.4	$1.000^{(60,1)}$ $1.104^{(60,2)}$	$1.946^{(60,3)}$ $2.125^{(60,4)}$	$2.886^{(60,5)}$ $3.143^{(60,6)}$			0.019
1	0.96	373.4	$1.000^{(180,1)}$ $1.027^{(180,2)}$ $1.050^{(180,3)}$ $1.070^{(180,4)}$ $1.089^{(180,5)}$	$1.973^{(180,6)}$ $2.007^{(180,7)}$ $2.035^{(180,8)}$ $2.060^{(180,9)}$ $2.083^{(180,10)}$	$2.944^{(180,11)}$ $3.043^{(180,12)}$ $3.069^{(180,13)}$	$3.913^{(180,14)}$ $4.051^{(180,15)}$		0.021
3	1.06	129.4	$1.000^{(60,1)}$ $1.056^{(60,2)}$	$1.946^{(60,3)}$ $2.015^{(60,4)}$ $2.073^{(60,5)}$ $2.125^{(60,6)}$	$2.965^{(60,7)}$ $3.143^{(60,8)}$	$3.824^{(60,9)}$ $4.045^{(60,10)}$	$4.760^{(60,11)}$ $5.060^{(60,12)}$	0.016
-	-	2.405	$1.000^{(0,1)}$	$1.938^{(1,1)}$	$2.949^{(2,1)}$ $3.059^{(0,2)}$	$3.967^{(3,1)}$ $4.114^{(1,2)}$	$4.877^{(0,3)}$ $5.023^{(4,1)}$ $5.178^{(2,2)}$	0.004

Applying (24) to a uniform density membrane, i.e. for $\theta^* = 0$ and $\sigma = 1$, gives 0.1369. Since our first aim is to characterize circular membranes having a degree of harmony higher than that of a uniform density membrane, hereafter we shall focus on membranes whose distinct angular sector has physical parameters θ^* and σ such that $E(\theta^*, \sigma) < 0.1$.

The graph of Fig. 2 shows the values of $E(\theta^*, \sigma)$ for the above 23 values of θ^*, and all values of $\sigma \in [0.1, 3]$ covered with a step size of 0.01. Table 1 gives the values of θ^* and σ corresponding to the lowest values of $E(\theta^*,\sigma)$ for vibrations with one to five consecutive quasi-harmonics ω_1, ω_2, ω_3, ω_4 and ω_5. The configurations are thus ordered according to the increasing value of the timbre.

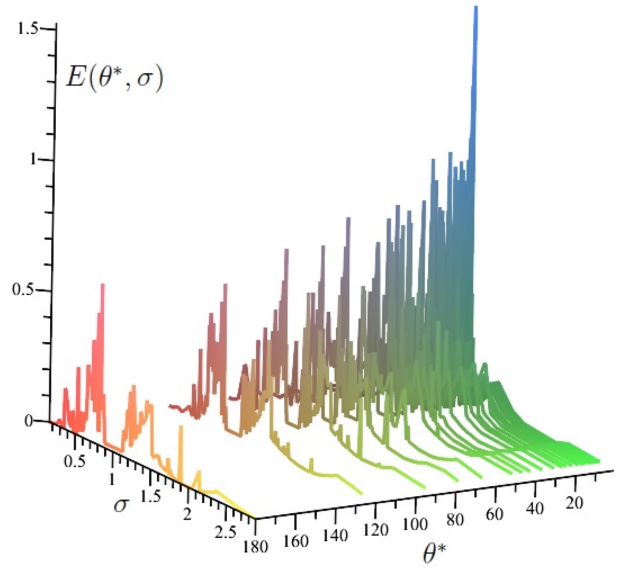

Figure 2. Graph of $E(\theta^*,\sigma)$, where θ^* is any integer divisor of 360 different from 360, and $\sigma \in [0.1, 3]$.

Any number in its first column can be paired with any number in the second column, belonging to the same horizontal segment, to form a configuration (θ^*, σ). This notification also applies to tables 2 and 5. In the first column, 1-180 stands for the set of angles $\theta^*=360/\ m_*$ where m_* is any integer divisor of 360 different from 1. The pair of upper indices (d, c) applied to numbers in the columns of ω_j describes the vibration mode of this normalized angular frequency, where d is the number of nodal diameters and c is the number of nodal circles. We obtain the normalized angular frequencies appearing in columns ω_1 to ω_5 through the division by Λ of the corresponding zeros of $J_{lm*}(x)$ that satisfy (23).

By way of comparison, the last horizontal segment of this table contains the relevant characteristics applying to the membrane of a tabla. Table 2 displays the characteristics of the best configurations allowing the membrane to vibrate with two to four nonconsecutive quasi-harmonics. The configurations are presented according to the increasing values of E. Observe that Λ = 2.405 also corresponds to the fundamental frequency of uniform density membranes, and that large values for Λ implies that the membrane vibrations will be composed of modes with very high frequencies.

Table 2. Characteristics of membranes with one distinct angular sector showing two to four nonconsecutive quasi-harmonics, ordered according to the increasing values of E.

θ^*	σ	Λ	ω_1	ω_2	ω_3	ω_4	ω_5	E
120	0.32 0.33	6.380	$1.000^{(3,1)}$		$3.042^{(3,2)}$			0.000
1-180	0.20 0.21	2.405	$1.000^{(180,1)}$				$4.903^{(180,2)}$	0.000
1-120	0.27 0.28	2.405	$1.000^{(180,1)}$			$3.598^{(180,2)}$		0.010
180	0.27	2.405	$1.000^{(1,1)}$			$3.598^{(1,2)}$		0.010
180	0.28	2.405	$1.000^{(1,1)}$	$2.135^{(1,2)}$		$3.598^{(1,3)}$		0.015
45 90	0.46 0.47 2.14	5.520	$1.000^{(4,1)}$	$2.136^{(4,2)}$ $2.215^{(4,3)}$			$4.758^{(4,4)}$	0.019
180	0.44	2.405	$1.000^{(1,1)}$	$2.135^{(1,2)}$ $2.295^{(1,3)}$	$3.155^{(1,4)}$		$4.832^{(1,5)}$	0.030

3. MEMBRANES WITH *N* EQUAL ANGULAR SECTORS

By extending the method of the previous section, we shall now determine the configurations that minimize the disharmony of a circular membrane of unit radius formed by N sectors having the same central angle of 360/N degrees (see Fig. 3). Note that to be distinct contiguous sectors are assumed to have different densities.

3.1. Mathematical model and numerical method

Let $\psi_k(r,\theta,t)$, k = 0, 1, …, $N - 1$, denote the functions describing the movement of the N sectors relative to their positions at rest. Following the lines of the mathematical modelling of Section 2, it is straightforward to show that

$$\partial_t^2\ \psi_k = c_k^2(\partial_r^2\ \psi_k + \frac{1}{r}\partial_r\ \psi_k + \frac{1}{r^2}\partial_\theta^2\ \psi_k), \qquad (25)$$

where $k = 0,1, \ldots, N-1$, $c_k^2 = \tau_k / \rho_k$ and $\tau_k = T_k / h_k$ in which T_k is the radial tension applied to the circular contour and h_k is the thickness of sector k. The equations expressing what happens at the junctions between the sectors are

$$\psi_k(r,(2\pi/N)((k+1) \bmod N),t) = \psi_{(k+1) \bmod N}(r,(2\pi/N)((k+1) \bmod N),t), \quad (26)$$

$$\partial\theta\psi k(r,(2\pi/N)((k+1) \bmod N),t) = \partial\theta\psi(k+1) \bmod N (r,(2\pi/N)((k+1) \bmod N),t), \quad (27)$$

where $k = 0,1, \ldots, N-1$ and $r \neq 0$ in (27). We also have the boundary conditions $\psi_k(1,\theta,t) = 0$ for $2\pi k/N \le \theta \le 2\pi(k+1)/N$, $k = 0,1, \ldots, N-1$.

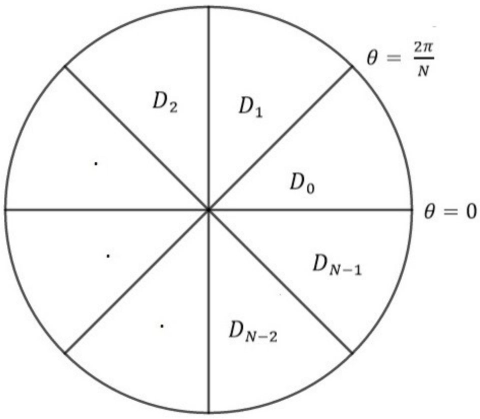

Figure 3. Schematic of a circular membrane with N equal angular sectors.

The substitution of $\psi_k(r,\theta,t) = R_k(r)\Theta_k(\theta)T_k(t)$ into (25) leads to the same expressions (12), (13) and (14) for $T_k(t)$, $\Theta_k(\theta)$ and $R_k(r)$, $k = 0,1, \ldots, N-1$. The approach used in Section 2, allows us here to show that for the membrane vibrations to be periodic in space the μ_k in (13) and (14) must be equal and given by lN where $l \in \boldsymbol{N}$. Moreover, the vibrations will be periodic in time if the values of ρ_k, $k = 0,1, \ldots, N-1$, are such that

$$\lambda^{(0)} c_0 = \lambda^{(1)} c_1 = \cdots = \lambda^{(N-1)} c_{N-1}, \quad (28)$$

where the $\lambda^{(k)}$ are distinct zeros of $J_{lN}(x)$.

For any set of $\rho_0, \rho_1, \cdots, \rho_{N-1}$ it is impossible to satisfy (28) for all sets of N-tuples of zeros of $J_{lN}(x)$. In practice however, it is enough to consider the sets of zeros related to the most energic modes of vibration of the membrane. Therefore, as in Section 2, we restrict ourselves to the orders of $J_{lN}(x)$ leading to frequencies near the fundamental and its first four harmonics.

Let M be an integer at least equal to the number of distinct densities among $\rho_0, \rho_1, \ldots, \rho_{N-1}$. For any given value of N, we suppose that amid the first $5(M-1)$ zeros of $J_{lN}(x)$, there are M-tuples of distinct zeros $\lambda_{lN,i}^{(0)}, \lambda_{lN,i}^{(1)}, \cdots, \lambda_{lN,i}^{(M-1)}$, $i = 1, 2, \ldots, I$, satisfying

$$\left| \frac{\lambda_{lN,i}^{(j_1)}}{\lambda_{lN,i}^{(j_2)}} - \sqrt{\frac{\rho_{j_1}}{\rho_{j_2}}} \right| < 0.01, \quad (29)$$

for all permutations of $j_1, j_2 \in \{0, 1, \ldots, M-1\}$. The angular frequency assigned to each of these M-tuples is defined by $\omega_{lN,i} = \lambda_{lN,i}^{(0)} c_0$ for $i = 1, 2, \ldots, I$. By doing this for $l = 0, 1, 2, 3, 4, 5$, we find a set of angular frequencies $\{\omega_{m_l i}\}$. Let Λ be the smallest zero of the form $\lambda_{lN,i}^{(0)}$ of the Bessel function associated with a given N and satisfying (29) for the set of values of $\sqrt{\rho_{j_1}/\rho_{j_2}}$, $j_1, j_2 \in \{0, 1, \ldots, M-1\}$. Then Λ corresponds to the mode of vibration that determines the fundamental angular frequency Ω of the membrane, which here again is given by $\Omega = \Lambda c_0$.

The global deviation from the first five harmonics is here defined by

$$E(N, \rho_0, \rho_1, \cdots, \rho_{N-1}) = \sum_{h=1}^{5} \left(\frac{\lambda_{lN,i}^{(0)}}{h\Lambda} - 1 \right)^2, \quad (30)$$

where the summation is for all zeros $\lambda_{lN,i}^{(0)}$ of $J_{lN}(x)$ that are related to the frequencies contributing to the harmonics of ranks 1 to 5. As in Section 2, any $\lambda_{lN,i}^{(0)}$ such that $\lambda_{lN,i}^{(0)} / \Lambda \in [h - 1/2, h + 1/2[$ is associated with $h \in \{1, 2, 3, 4, 5\}$. One easily sees that once N and the number of different densities are given, then this method is independent of the way the densities are distributed over the membrane sectors. Consequently, if there are more than one such distribution, the results will be applicable to each of these. The values of N and $\rho_0, \rho_1, \ldots, \rho_{N-1}$ minimizing (30) will characterize the configurations with minimal disharmony.

3.2. Membranes with two or three different densities

Let us now apply the above method to membranes with two or three different densities over

N angular sectors of $360/N$ degrees. For membranes with two densities, it is easily seen that N must be an even integer and that (29) reduces to (23). The solution of the present problem is then the same as the one solved in Section 2 for even values of m. The optimal configurations are thus those associated with the 18 values of θ^* given by 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90 and 180.

We now consider membranes having three different densities ρ_0, ρ_1 and ρ_2 on angular sectors of the same integer number of degrees.

The possibilities for *N* are here the integer divisors of 360, from which we remove $N = 1$ and $N = 2$ because it is impossible to distribute three different densities on a membrane having only one or two sectors.

This leaves us with 22 possibilities for *N*. If we set $\sigma_1 = \sqrt{\rho_0/\rho_2}$ and $\sigma_2 = \sqrt{\rho_0/\rho_1}$, then (29) reduces to

$$\left|\frac{\lambda_{lN,i}^{(0)}}{\lambda_{lN,i}^{(1)}} - \sigma_2\right| < 0.01, \left|\frac{\lambda_{lN,i}^{(0)}}{\lambda_{lN,i}^{(2)}} - \sigma_1\right| < 0.01, \quad \left|\frac{\lambda_{lN,i}^{(1)}}{\lambda_{lN,i}^{(2)}} - \frac{\sigma_1}{\sigma_2}\right| < 0.01, \tag{31}$$

Since $\rho_1 \neq \rho_2$, without lost of generality we can assume that $\sigma_1 < \sigma_2$.

By giving to each component of (σ_1, σ_2), $\sigma_1 < \sigma_2$, the values in the interval [0.1, 3] covered with a step size of 0.01, we determine triplets $(\lambda_{lN,i}^{(0)}, \lambda_{lN,i}^{(1)}, \lambda_{lN,i}^{(2)})$ whose components satisfy (31).

When checking if three zeros λ^*, λ^{**} and λ^{***} of $J_{lN}(x)$ satisfy (31), we must consider separately the six different triplets these zeros can form, because it is possible that one of these does it while all other do not. Moreover, it is straightforward to show that when applying the test provided by (31), we can limit ourselves to triplets $(\lambda^*, \lambda^{**}, \lambda^{***})$ such that $\lambda^{**} < 1.01\,\lambda^{***}$.

The angular frequency associated with the triplet $(\lambda_{lN,i}^{(0)}, \lambda_{lN,i}^{(1)}, \lambda_{lN,i}^{(2)})$ is defined by $\omega_{lN,i} = \lambda_{lN,i}^{(0)} c_0$ for $i = 1, 2, \ldots, I$. Here (30) becomes

$$E(N, \sigma_1, \sigma_2) = \sum_{h=1}^{5}\left(\frac{\lambda_{lN,i}^{(0)}}{h\Lambda} - 1\right)^2.$$

Table 3 gives the values of $\theta^* = 360/N$, σ_1 and σ_2 corresponding to the lowest values of $E(N, \sigma_1, \sigma_2)$ for vibrations with one to five consecutive quasi-harmonics ω_1, ω_2, ω_3, ω_4 and ω_5. We obtain the values appearing in columns of ω_1 to ω_5 through the division by Λ of the corresponding zeros of $J_{lN}(x)$.

Table 4 presents the characteristics of the best membrane configurations allowing the membrane to vibrate with two to four non-consecutive quasi-harmonics, ordered according to the increasing values of *E*.

Table 3. Characteristics of three-density membranes with N=360/θ^* distinct equal angular sectors showing the lowest values of *E* and one to five consecutive quasi-harmonics. To form a configuration (θ^*, σ_1, σ_2), each number σ_1 must be paired with the number on the same line in the column of σ_2. This notification also applies to tables 4 and 6.

θ^*	σ_1	σ_2	Λ	ω_1	ω_2	ω_3	ω_4	ω_5	E
2	1.00	1.02	942.9	$1.000^{(90,1)}$ $1.011^{(90,2)}$ $1.020^{(90,3)}$					0.001
1	0.97 1.03	1.03 1.06	383.5	$1.000^{(180,1)}$ $1.022^{(180,2)}$ $1.060^{(180,3)}$	$1.921^{(180,4)}$ $1.981^{(180,5)}$ $2.028^{(180,6)}$				0.006
120	0.33 0.67 0.67 0.68 0.68	0.49 2.03 2.04 2.03 2.04	6.380	$1.000^{(3,1)}$	$2.040^{(3,2)}$	$3.042^{(3,3)}$			0.001
1	0.96	0.97	383.5	$1.000^{(180,1)}$ $1.022^{(180,2)}$ $1.042^{(180,3)}$	$1.921^{(180,4)}$ $1.954^{(180,5)}$ $1.981^{(180,6)}$ $2.006^{(180,7)}$ $2.028^{(180,8)}$	$2.866^{(180,9)}$ $2.935^{(180,10)}$ $2.962^{(180,11)}$ $2.988^{(180,12)}$	$3.810^{(180,13)}$ $3.915^{(180,14)}$ $3.943^{(180,15)}$		0.010
1	1.01	1.04	383.5	$1.000^{(180,1)}$ $1.022^{(180,2)}$ $1.042^{(180,3)}$ $1.060^{(180,4)}$	$1.921^{(180,5)}$ $1.954^{(180,6)}$ $1.981^{(180,7)}$ $2.006^{(180,8)}$ $2.028^{(180,9)}$	$2.866^{(180,10)}$ $2.935^{(180,11)}$ $2.962^{(180,12)}$ $2.988^{(180,13)}$	$3.810^{(180,14)}$ $3.885^{(180,15)}$ $3.915^{(180,16)}$ $3.943^{(180,17)}$	$4.752^{(180,18)}$ $4.866^{(180,19)}$ $4.896^{(180,20)}$	0.018

Table 4. Characteristics of three-density membranes with N=360/θ* distinct equal angular sectors showing two to four nonconsecutive quasi-harmonics, ordered according to the increasing values of E.

θ^*	σ_1	σ_2	Λ	ω_1	ω_2	ω_3	ω_4	ω_5	E
1-120	0.16	0.28	2.405	$1.000^{(180,1)}$			$3.599^{(180,2)}$		0.010
1-120	0.20	0.27	2.405	$1.000^{(180,1)}$			$3.599^{(180,2)}$	$4.903^{(180,3)}$	0.010
1-120	0.20 0.21 0.46 0.46 0.47 0.47	0.43 0.44 2.29 2.30 2.29 2.30	2.405	$1.000^{(180,1)}$	$2.295^{(180,2)}$			$4.903^{(180,3)}$	0.022

4. EASY TO BUILD MUSICAL CONFIGURATIONS

The algorithms applied in Sections 2 and 3 yield optimal configurations of membranes with angular sectors of different densities. However, some of these configurations might be hard to realize physically. This would probably be the case, for instance, if θ* is very small, or if σ, or σ_1 and σ_2, are very close to 1. Other configurations are not musically interesting, due to their very high fundamental frequency. In Tables 5 and 6, we present some theoretically easy to build configurations which, without being optimal, would produce vibrations with musical qualities notably superior to those of uniform density membranes.

Table 5. Characteristics of easy to build membranes with one distinct angular sector producing small fundamental frequency. The configurations are ordered according to the decreasing number of quasi-harmonics.

θ^*	σ	Λ	ω_1	ω_2	ω_3	ω_4	ω_5	E
120	0.63	5.520	$1.000^{(3,1)}$	$1.568^{(3,2)}$	$3.123^{(3,3)}$	$3.622^{(3,4)}$	$4.997^{(3,5)}$	0.057
180	0.43	2.405	$1.000^{(1,1)}$	$2.295^{(1,2)}$	$3.156^{(1.3)}$	$4.132^{(1,4)}$		0.026
90	0.43 0.44	2.405	$1.000^{(2,1)}$	$2.295^{(2,2)}$	$3.156^{(2,3)}$			0.025
180	0.56	8.417	$1.000^{(1,1)}$	$1.720^{(1,2)}$ $1.758^{(1,3)}$	$3.031^{(1,4)}$			0.034

Table 6. Characteristics of easy to build membranes with N=360/θ* distinct equal angular sectors of three different densities producing small fundamental frequency. The configurations are ordered according to the decreasing number of quasi-harmonics.

θ^*	σ_1	σ_2	Λ	ω_1	ω_2	ω_3	ω_4	ω_5	E
120	0.49	0.65	6.380	$1.000^{(3,1)}$	$1.530^{(3,2)}$ $2.040^{(3,3)}$ $2.093^{(3,4)}$	$3.261^{(3,5)}$	$4.323^{(3,6)}$		0.072
120	0.33 0.67 0.67 0.68 0.68	0.49 2.03 2.04 2.03 2.04	6.380	$1.000^{(3,1)}$	$2.040^{(3,2)}$	$3.042^{(3,3)}$			0.001
90	0.36 0.37 0.69 0.69 0.70 0.70 1.44 1.44 1.45 1.45	0.52 0.53 1.89 1.90 1.89 1.90 2.74 2.75 2.74 2.75	7.588	$1.000^{(2,1)}$	$1.894^{(2,2)}$	$2.745^{(2,3)}$			0.010

5. CONCLUSIONS

We have formulated and exemplified a mathematical method to minimize the disharmony of a circular membrane whose density differs over two complementary angular sectors, or N angular sectors of 360/N degrees. From a close examination of the various numerical results we can infer that if the membrane has many sectors of the same angle, then the fundamental frequency of its vibration will generally be high. Conversely, if its sectors are few, then the fundamental frequency of its vibrations will be low. The area of the sectors is thus inversely proportional to the fundamental frequency of vibration. This mathematical observation goes with certain natural phenomena. An example of these is the frequency of the regular beat of bird wings, seen as fundamental frequency, which is inversely proportional to the area of the wings.

Concerning the musical qualities of such circular membranes, we got optimal configurations that minimize the global deviation of the membrane natural modes of vibration from the first five harmonics. However, some of these configurations might be hard to realize physically, especially those involving many small angular sectors, or very small differences between the densities of its sectors. Others may be of little musical interest due to their poor timbre or their very high fundamental frequencies. Leaving aside these outcomes without obvious musical consequence, we put forward in Section 4 the characteristics of new musical configurations that should possibly be easy to build. Even if not optimal, these are likely to generate vibrations having better musical qualities than those of uniform density membranes. When comparing the timbre and the value of E of each of the configurations presented in this paper with those of the tabla configuration shown in the last horizontal segment of Table 1, we see that the latter has better musical qualities. Nevertheless, membranes with any configuration picked among those of Tables 5 or 6 are likely to produce sounds with better musical qualities than any regular membrane, with the added benefit of their original design.

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