A Review of Interpolation Methods Used for Frequency Estimation

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Abstract: - Frequency estimation for short-time signals is difficult because the raw frequency resolution obtained. The frequency for these signals fall usually at an inter-line position, and is far from the frequency displayed at the spectral line. To get a better frequency estimate, interpolation methods are used to find the peak between the two spectral lines on which the biggest amplitudes are obtained. The number of points involved in the interpolation are two or three. The inter-line position at which this maximum is found is considered to correspond to the true frequency. This paper presents a comparison of the most common interpolation methods that can be used to increase the precision of the frequency estimation, so necessary in the processes of structural health and condition monitoring. An analysis of the accuracy obtained by these methods is made for a generated signal with an acquisition time of around 1 second.

Keywords: - Discrete Fourier Transform, frequency estimation, interpolation methods, vibration signals

1. INTRODUCTION

The problem of accurately estimating frequencies of signals is important in different applications, such us communication [1], radar and sonar [2], biomedicine [3] etc. In vibration analysis, a typical problem consists of estimating the frequencies changes of structures or machinery to find the development of a fault [4]. If long data records are available, the discrete Fourier transforms (DFT) provides enough resolution for frequency estimation and operates in real-time. The fast Fourier transform (FFT) is an algorithm from the DFT, which requires even less computational effort and is, therefore, faster. However, it was shown that both the DFT and the FFT have serious limitations in providing accurate results for short-time signals [5].

Various methods for improving frequency readability based on interpolation or windowing have been presented in the literature. Thus, a series of frequency estimation methods are proposed, which are based on the use of two or three points from the DFT complex values of spectrum [6]-[9] or from the module spectrum [10]-[13]. Comparative studies were performed between different methods that allow accurate estimation of frequencies and their effectiveness was analyzed [5], [8], [14]. There are also methods that consider a trim-to-fit strategy [15],[16] or zero-padding [17], which is also an interpolation method. In our works, we used the frequencies to detect damage; an accurate estimation permits an early crack identification [18], [19].

In this paper we present a review of the most common interpolation methods employed to estimate the frequencies of signals, along with a comparison of the results obtained for generated signals in order to test their efficiency.

2. INTERPOLATION METHODS

Let us consider a sinusoidal signal as a sampled sequence

$$a[n] = A\exp\left[j2\pi \frac{f_0}{f_S} n \right], n = 0,...,N-1 \qquad (1)$$

where A is the amplitude, f_0 is the frequency of the signal, f_S is the sampling frequency, N the number of acquired samples and n the index of the samples and $j^2 = -1$.

The frequency resolution:

$$\Delta f = \frac{f_S}{N} \qquad (2)$$

is the distance between two spectral lines.

The discrete Fourier transform DFT of $a[n]$ at spectral line k displays the amplitude:

$$A_k = \sum_{n=0}^{N-1} a[n] e^{-j(2\pi/N)nk} \quad (3)$$

which has the form $A_k = \mathrm{Re}\,A_k + j\,\mathrm{Im}\,A_k$.

The real and imaginary coefficients have the form:

$$\mathrm{Re}\,A_k = \sum_{n=0}^{N-1} a[n] \cos\left(\frac{2\pi}{N} nk\right) \quad (4)$$

$$\mathrm{Im}\,A_k = -\sum_{n=0}^{N-1} a[n] \sin\left(\frac{2\pi}{N} nk\right) \quad (5)$$

The modulus of A_k is calculated with the formula:

$$|A_k| = \sqrt{(\mathrm{Re}\,A_k)^2 + (\mathrm{Im}\,A_k)^2} \quad (6)$$

In order to determine the true frequency, two stages are followed. In the first stage an index k of DFT is sought for which $|A_k|$ is maximum, and in the second stage we will look for a δ (a correction coefficient) in the vicinity of k so that the estimated (corrected) frequency is:

$$f_c = (k + \delta)\Delta f \quad (7)$$

To find out this correction factor, different methods are used that interpolate DFT coefficients in the vicinity of the maximizer A_k.

There are two categories of methods, classified as follows:

- methods depending on the number of points considered for interpolation: two and three-point methods;

- depending on the type of involved data: based on the complex coefficients and based on the spectrum of modules.

We exemplify these methods according to the way in which their authors position them in the above classification (Table 1).

Table 1. Classification of interpolation methods

Type of points for analysis	Two-point methods	Three-point methods
Real part methods	Quinn [6] Grandke [10]	Jacobsen [7]
Module-based methods	Jain et al. [11]	Ding [12] Voglewede [13]

In the case of methods that use three points for interpolation, the maximizer A_k and its neighbors A_{k-1} and A_{k+1} are selected for the interpolation, while in the case of the algorithms based on two points, the maximizer and its largest neighbor are chosen.

The selection of the larges neighbor will be proved to be a mistake, because not always the biggest neighbor is on the main lobe.

Figure 1 represents the real coefficients in the Fourier series and Figure 2 the imaginary coefficients. When using methods based on the real part, one of the neighbors is negative, as shown in Figure 1. It is also possible to obtain ImA_k with a negative value. So, after choosing the largest A_k in DFT, the real and imaginary parts for the maximizer and its neighbors are selected.

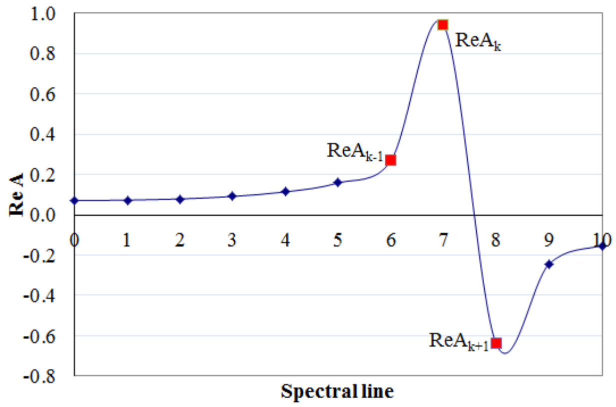

Figure 1. The three points in the real part of the spectrum used for interpolation

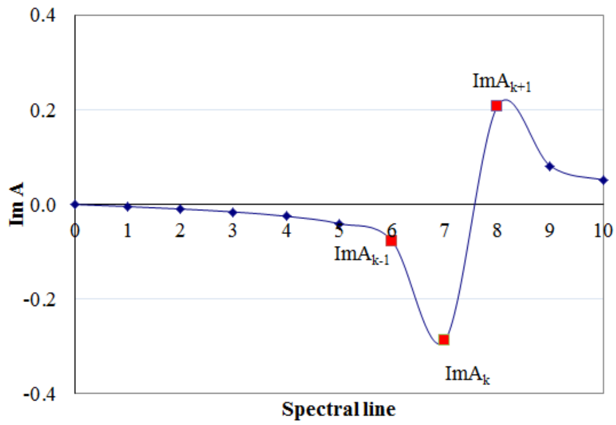

Figure 2. The three points in the imaginary part of the spectrum used for interpolation

Regarding the interpolation methods based on modules, they use the DFT spectrum, from which they extract $|A_k|$ and points in the vicinity of this maximizer. In the DFT spectrum (Figure 3) it is observed that not all three points are on the main lobe. Moreover, there are cases where the biggest neighbor of the maximizer is not on the main lobe.

It's needed to find the maximum of the curve that passes through these three chosen points and thus we can determine the corrected frequency with the help of a fractional correction term δ.

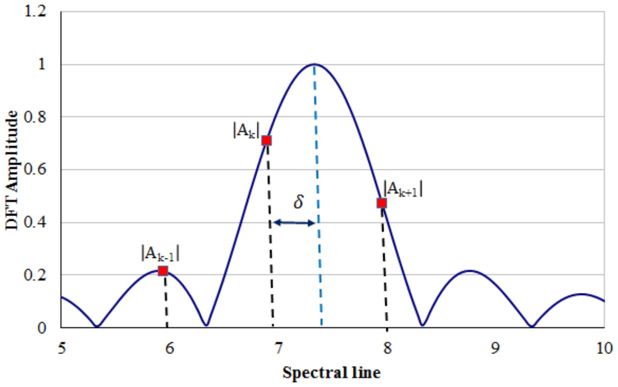

Figure 3. The three points in the DFT spectrum used for interpolation

The correction coefficient and the formula according to which the corrected frequency is calculated have different versions, depending on the number of points in the spectrum that are used for interpolation.

In the next sections, we make an analysis of the accuracy of the interpolation methods developed by the authors mentioned in Table 1.

2.1. Module-based methods

The frequency calculation programs in the dedicated software give us information about FFT, DFT, PS and PSD, in all being calculated and displayed the spectrum of the modules. For this reason, they are the handiest.

In [11], Jain uses 2 points that are in the spectrum for modules. The method developed by Jain is based on the following algorithm:

If $|A_{k-1}| > |A_{k+1}|$, then $\alpha_1 = |A_k| / |A_{k-1}|$ and the correction term $\delta_1 = \alpha_1 / (1+\alpha_1)$ results. The frequency is estimated with the mathematical formula:

$$f_c = (k - 1 + \delta_1)\Delta f \qquad (8)$$

else, $\alpha_2 = |A_{k+1}| / |A_k|$ and $\delta_2 = \alpha_2 / (1+\alpha_2)$, resulting the frequency:

$$f_c = (k + \delta_2)\Delta f \qquad (9)$$

From the interpolation methods that involve three points we chose to analyze the algorithms proposed by Ding [9] and Voglewede [10], these being some methods based on modules of the maximizer A_k and its neighbors located on the spectral lines *k-1* and *k+1*, respectively.

Ding propose, for the correction coefficient the relation:

$$\delta = \frac{|A_{k+1}| - |A_{k-1}|}{|A_{k-1}| + |A_k| + |A_{k+1}|} \qquad (10)$$

while Voglewede use the relation:

$$\delta = \frac{|A_{k+1}| - |A_{k-1}|}{2(2|A_k| - |A_{k-1}| - |A_{k+1}|)} \qquad (11)$$

The corrected frequency is further determined with the formula (9) in which the correction coefficient is replaced by (10) or (11), depending on the method used. Numerical examples for estimated frequencies using these three algorithms are given in section 3.

2.2. Real part methods

Among the interpolation methods that involve calculating the real part of a fraction, we chose to analyze the algorithms proposed by Quinn [6] and Jacobsen's formula [7].

The algorithm developed by Quinn requires the choice of three points with the amplitude A_k its neighbors A_{k-1} and A_{k+1}, but depending on the corresponding correction coefficient, in determining the corrected frequency will appear only the amplitudes A_{k-1} and A_k if the correction coefficient is δ_1 or A_{k+1} and A_k, if the coefficient is δ_1. Quinn proposed the calculation of the coefficients α_1, α_2 and depending on the nature of the correction coefficients δ_1 and δ_2 the corrected frequency according to formula (12) is calculated.

$$\alpha_1 = \mathrm{Re}\left\{\frac{A_{k-1}}{A_k}\right\}, \ \delta_1 = \frac{\alpha_1}{1-\alpha_1} \qquad (12)$$

$$\alpha_2 = \mathrm{Re}\left\{\frac{A_{k+1}}{A_k}\right\}, \ \delta_2 = \frac{-\alpha_2}{1-\alpha_2} \qquad (13)$$

If $\delta_1 > 0$ and $\delta_2 > 0$ then $\delta = \delta_2$, else $\delta = \delta_1$.

$$f_c = (k + \delta)\Delta f \qquad (14)$$

To determine α_1 and α_2 we need to calculate the real part of a fraction, which actually involves determining the real part and the imaginary part for each A_k, A_{k-1} and A_{k+1}, values that allow determining these coefficients.

$$\alpha_2 = \mathrm{Re}\left\{\frac{A_{k-1}}{A_k}\right\} = \frac{\mathrm{Re}A_{k-1}\cdot\mathrm{Re}A_k + \mathrm{Im}A_{k-1}\cdot\mathrm{Im}A_k}{|A_k|^2} \quad (15)$$

$$\alpha_2 = \mathrm{Re}\left\{\frac{A_{k+1}}{A_k}\right\} = \frac{\mathrm{Re}A_{k+1}\cdot\mathrm{Re}A_k + \mathrm{Im}A_{k+1}\cdot\mathrm{Im}A_k}{|A_k|^2} \quad (16)$$

where $A_k = \mathrm{Re}A_k + i\,\mathrm{Im}A_k$, $A_{k+1} = \mathrm{Re}A_{k+1} + i\,\mathrm{Im}A_{k+1}$ and $A_{k-1} = \mathrm{Re}A_{k-1} + i\,\mathrm{Im}A_{k-1}$.

The correction coefficient proposed by Jacobsen [7] is:

$$\delta = \mathrm{Re}\left\{\frac{A_{k-1} - A_{k+1}}{2A_k - A_{k-1} - A_{k+1}}\right\} \quad (17)$$

or written in an explicit form:

$$\delta = \frac{(\mathrm{Re}\,A_{k-1} - \mathrm{Re}\,A_{k+1})(2\,\mathrm{Re}\,A_k - \mathrm{Re}\,A_{k-1} - \mathrm{Re}\,A_{k+1})}{S} + \\ + \frac{(\mathrm{Im}\,A_{k-1} - \mathrm{Im}\,A_{k+1})(2\,\mathrm{Im}\,A_k - \mathrm{Im}\,A_{k-1} - \mathrm{Im}\,A_{k+1})}{S}$$

In the explicit form of equation (17), the denominator s takes the form:

$$S = (2\,\mathrm{Re}\,A_k - \mathrm{Re}\,A_{k-1} - \mathrm{Re}\,A_{k+1})^2 + \\ + (2\,\mathrm{Im}\,A_k - \mathrm{Im}\,A_{k-1} - \mathrm{Im}\,A_{k+1})^2 \quad (18)$$

With the help of the above formula, a fractional correction term δ is determined, so a corrected frequency f_c could be calculated with formula (14).

3. RESULTS AND DISCUTION

To exemplify how the algorithms work, we choose signals generated with known frequency and amplitude. Thus, we considered a signal with the frequency f_{true} = 7.33 Hz and the amplitude A_{true} = 1.

The signal is generated with a frequency rate r = 1000 samples/second.

The shortest signal time length is 0.84 seconds and is generated using 841 samples, while the longest considered signal time length is 1.235 seconds, being generated with 1236 samples. Between the two time limits, we generate signals by repetitively adding 5 samples to the previous signal. In some areas, where we need a finer representation, we generate signals for which the total number of samples is increased with 2 or even 1 sample by iteration.

The graph that represents the frequency estimation according to Jain's algorithm with formulas (8) or (9) is plotted in Figure 4 with a green line. In this figure it is also shown the generated frequency with a black line, to permit a facile evaluation of accuracy achieved when applying this algorithm.

The obtained results are generally good; it is observed that the estimated frequency achieves an absolute error in the range -0.26Hz … 0.26 Hz, which means a relative maximum error of 3.6%.

The correction coefficients proposed by Ding and Voglewede, with the help of which we estimated the frequencies and we represented them graphically in Figure 5, introduce a maximum error which is around 0.25 Hz (i.e. 3.6%). between the estimated and the real frequency.

In Figure 6 we graphically represent, with purple line, the corrected frequencies according to the formula given by Quinn, while with orange line we represent the frequencies obtained with the help of the correction coefficient given by Jacobsen. The representation is again made for the signal length in the 0.84…1.235 seconds.

For the frequency estimated according to the algorithm developed by Quinn the difference between this and the real frequency is about 0.015 Hz and the error is displaced around the true frequency value.

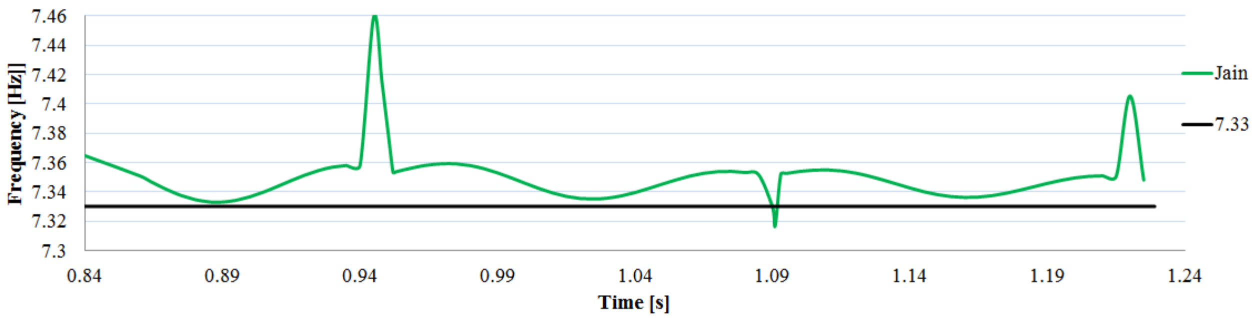

Figure 4. Frequency estimation according to Jain's algorithm

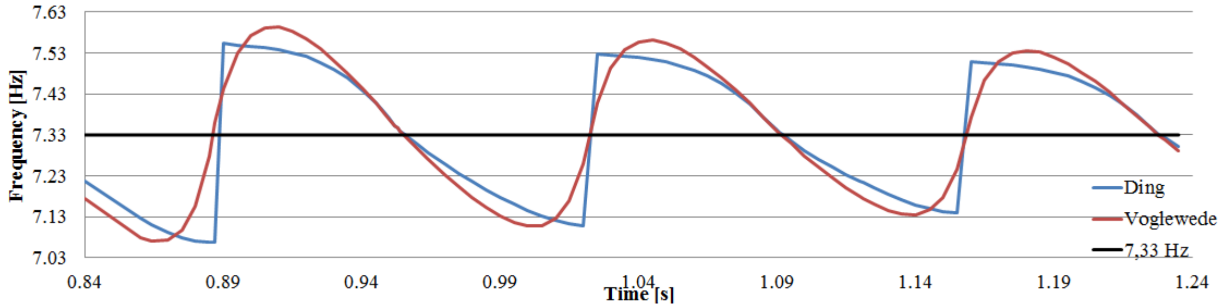

Figure 5. Frequency estimation according to Ding and Voglewede algorithms

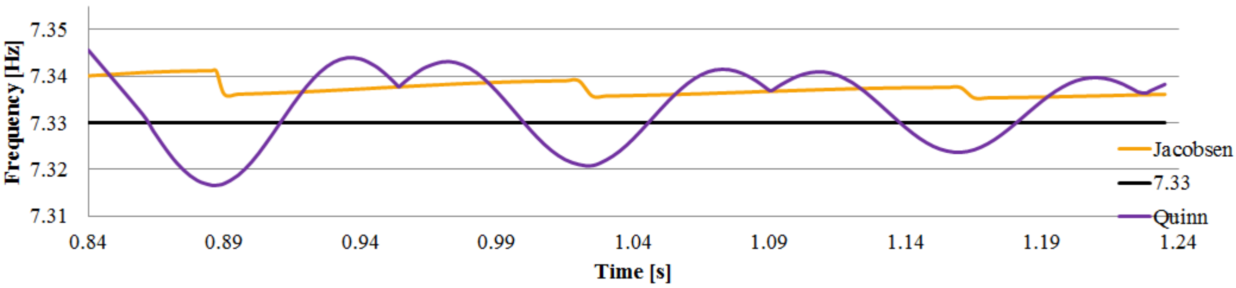

Figure 6. Frequency estimation according to Quinn and Jacobsen algorithms

In the case of estimating the frequency using the correction coefficient proposed by Jacobsen, this difference decreases below 0.01 Hz and the error is always positive (a frequency overestimation).

Analyzing the studied interpolation methods, it is found that when the signal contains an integer number of cycles, i.e. the time length t=1.09s, the algorithms developed by Ding and Voglewede allow obtaining the correct frequency. This is clearly demonstrated in Figure 7. Dissimilar, for this time length, and consequently for an entire number of cycles, the graph representing the frequency evolution plotted by using Jain's algorithm has a significant jump.

For the module based methods that involve three points, when the time is between 0,84s and 1.235s, the error varies between -3.6 % and 3.6% and for the method considering one maximizer neighbor, proposed by Jain, the deviation is a maximum of 1.8%.

Analyzing the frequency estimation accuracy of the methods based on the real part, it is found that the algorithm developed by Jacobsen overestimates the frequency by up to 0.15 Hz, while Quinn's algorithm brings a deviation of ±1% in the analyzed time interval.

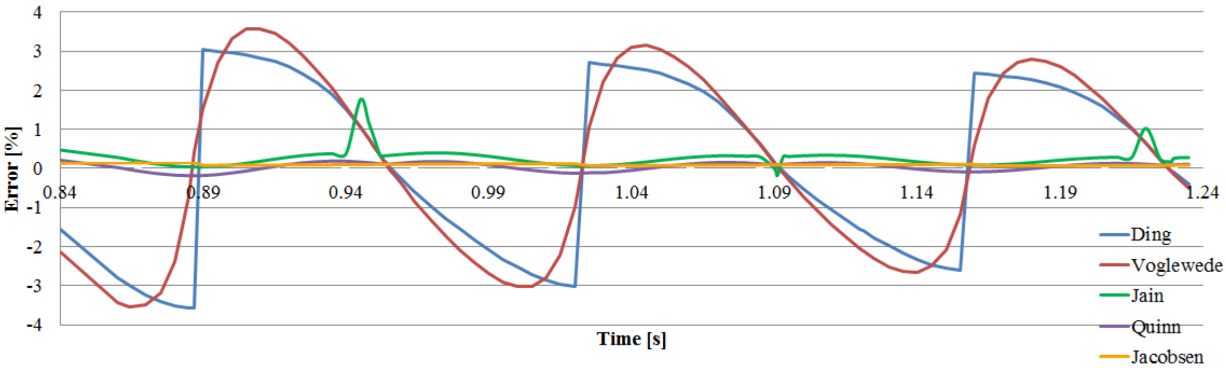

Figure 7. Errors that occur at frequency evaluation for the analyzed methods

In the results obtained according to Jain's algorithm we notice a jump up to 0.12 Hz when the time length of the analyzed signal is 0.958 seconds, (Figure 4). So, the error increases to 1.76%.

This happens because Jain propose to take $|A_{k+1}|$ and calculate α_2 and δ_2 to correct the frequency according to the formula (9).

Indeed, for t=0.948s we have $|A_{k-1}|<|A_{k+1}|$, but $|A_{k+1}|$ is not on the main lobe. In consequence, we propose using $|A_{k-1}|$ instead of $|A_{k+1}|$ because the first one is on the main lobe as $|A_k|$ is. So, we calculate α_1 and the resulted δ_1 for correcting the frequency.

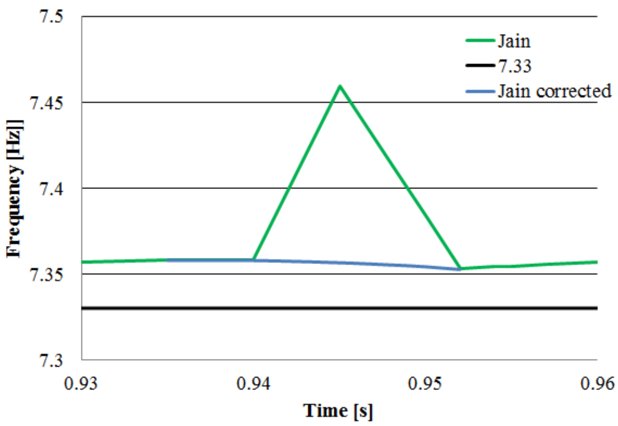

Figure 8. Frequencies estimated with the corrected Jain's algorithm

The results are significantly improved and the maximum error is reduced from 1.76% to 0.36%. For exemplification, the corrected frequency estimation curve is plotted in Figure 8 with a blue line and the original curve plotted using Jain's algorithm with a green line. The algorithm that can be involved to determine which neighbor is located on the main lobe is presented in [18].

4. CONCLUSIONS

The interpolation algorithms reviewed in this paper improve the frequency readability with different degrees of precision. It cannot be defined which type of algorithm leads to more accurate results, each of them improving the readability at different time lengths, i.e. for different fractions of time in the last cycle. However, the algorithms of Jacobsen and Kootsookos and that of Quinn seem to be the most reliable irrespective of the signal length. We also found that the analyzed algorithms can be improved, an example being presented for the algorithm of Jain, for which the maximum error was reduced five times.

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