A Mathematical Method for Harmonizing the Vibrations of Compound Cymbals

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Abstract: - Cymbal sounds are generally disharmonic. Using the property that their most energetic vibration modes are the same as those of thin circular plates, we develop a mathematical method to determine new configurations of compound cymbals made up of a centered disk and a ring around it that would vibrate with minimal disharmony. We exemplify the method with plates made of aluminum and copper.

Keywords: - quasi-harmonic vibrations, compound circular plates, cymbal, musical instruments

1. INTRODUCTION

The cymbals used by Western-style bands and orchestras are slightly tapered, with a small dome in the center. Chinese cymbals have a flattened central dome at the top and their edge is slightly turned. A hole in the center of the cymbal serves either to mount it on a support, or to attach a strap to play directly with the hands [1]. The nature of sounds produced by a cymbal varies according to its diameter, thickness, shape and the material of which it is made up. Cymbals are usually made of bronze.

Like most circular drum membranes, cymbals produce disharmonic sounds as a rule. This results from the property that their sounds do not break up into vibrations whose frequencies are integer multiples of a fundamental frequency. However, it is possible to make the sounds of a circular drum quasi-harmonic by using a membrane having more than one areal mass density. One such drum is the Indian tabla, whose membrane has a disk in its central region with a radius equal to 40% of that of the entire membrane and having a density about ten times that of the rest of the membrane [2, 3]. The purpose of this paper is to present a mathematical method to determine the values of physical parameters allowing the design of cymbals that should produce vibrations with minimal disharmony. From the musical point of view, cymbals of this kind should produce sounds having properties closer to harmonic musical instruments, such as stringed instruments. To reach this end, we shall adapt to cymbals the optimization method used in [4] to obtain new configurations of quasi-harmonic circular membranes with more than one density. Other adaptations of this method have also been used to theoretically design networks of strings [5], and circular membranes with different densities on angular sectors [6].

Our mathematical model of cymbals will be that of a thin circular plate. For the history of plate theory and a recent review of its applications, see e.g. [7, 8], and the references therein. This modeling is based on the fact that low-frequency vibration modes of a cymbal, which contain most of its vibration energy, are very similar to those of a thin circular plate [1]. As for circular membranes, these vibrations show diameters and circles of nodes. We designate by (m, n) its mode of vibrations having m nodal diameters and n nodal circles. When the cymbal is supported at its center, its modes of vibration always create at least one, or at least two nodal diameters. This property results from the physical characteristics of the cymbal, including the size of its central dome. For example, all modes of a 38 cm cymbal set up at least two nodal diameters [9], while all modes of a 46 cm crash cymbal build at least one nodal diameter [1]. In the modeling of a cymbal by a thin circular plate, we can therefore limit ourselves to vibrations having at least one, or at least two diameters of nodes. Our first interest here is not to solve the equations describing the movements of the plate but rather to obtain its eigenvalue equation, from which we shall deduce its eigenfrequencies. The values of the plate physical parameters determine the values of these eigenfrequencies. By varying the values of these physical parameters, we shall find out those allowing the plate, and therefore the cymbal, to vibrate with minimal disharmony.

The structure of this paper is as follows. The mathematical model of a thin circular plate made up of a centered disk and a ring around this disk is set up in Section 2. In Section 3, we present the equation used to characterize the eigenvalues of the two-region plate and explain how these eigenvalues are used to determine the physical parameters corresponding to cymbals with minimal disharmony. This eigenvalue equation is established in Appendix A. We exemplify the method by considering circular plates made of aluminum and copper in Section 4.

2. MATHEMATICAL MODEL

Let us consider the vibrations of a solid circular plate of unit radius having a centered disk of radius $0 < a \leq 1$ (see Fig. 1). This disk may have a different thickness or may be made of a material different from that of the rest of the plate. We designate by h_1 the thickness of the central disk, and by h_2 that of the ring around this disk, the values of both h_j being very small in front of the other dimensions of the plate. It is assumed that the plate is flat at rest and that its deformations remain weak, which means that the displacements, perpendicular to the plane of the plate, of each of its points are small compared to the two h_j. Using Hamilton's principle, it is straightforward to deduce the equations describing the motion of the plate, as well as the boundary and dynamical junction conditions along the circle $r = a$ (see e.g. [10] or [11]).

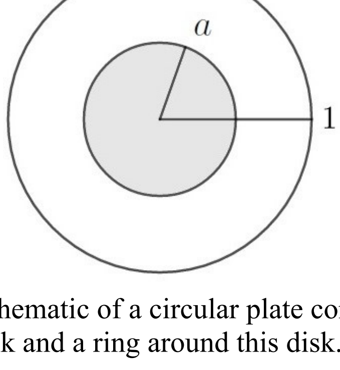

Figure 1. Schematic of a circular plate composed of disk and a ring around this disk.

To set out these equations, let $\psi(r,\theta,t)$ be a twice-differentiable function describing the small displacements at time t of the point of polar coordinate (r,θ) on the plate, with respect to its rest position. The domain of $\psi(r,\theta,t)$ is [0,1] × $\boldsymbol{R}$ × $\boldsymbol{R}_+$. Also, let $\psi_j(r,\theta,t)$, $j = 1, 2$, be twice-differentiable functions with the same domain as $\psi(r,\theta,t)$ and which coincide with the latter on the central disk $r \leq a$ if $j = 1$, and on the ring $a \leq r \leq 1$ if $j = 2$. The equations describing the plate vibrations are given by

$$c_j^4 \Delta^2 \psi_j + \frac{\partial^2 \psi_j}{\partial t^2} = 0, \quad j = 1, 2 \tag{1}$$

where Δ denotes the Laplacian, and

$$c_j^4 = \frac{E_j h_j^2}{12\rho_j\left(1-\nu_j^2\right)}, \tag{2}$$

whereas E_j, h_j, ρ_j and ν_j are Young's modulus, thickness, mass density and Poisson's coefficient, respectively, of the materials in regions $j = 1, 2$ of the plate.

The assumed periodicity in time of vibrations implies that the dependence in t of each function ψ_j can be separated from those in r and θ. Therefore

$$\psi_j(r,\theta,t) = \varphi_j(r,\theta)\, T_j(t), \quad j = 1, 2, \tag{3}$$

where φ_j are functions of $r \in [0,1]$ and $\theta \in \boldsymbol{R}$ only, whereas the functions T_j depend only upon $t \in \boldsymbol{R}_+$. Substituting (3) into (1) directly leads to

$$T_j''(t) + \lambda_j^4\, c_j^4 T_j(t) = 0, \tag{4}$$

and

$$\frac{\partial^2 \varphi_j(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_j(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi_j(r,\theta)}{\partial \theta^2} \pm {\lambda_j}^2 \varphi_j(r,\theta) = 0, \tag{5}$$

where λ_j^4, $j = 1, 2$, are separation constants.

We now formulate the equations related to the conditions applying at the boundaries of the plate and those at the junction between the central disk and its surrounding ring. Let us first consider the boundary conditions applying at the center of the plate, and along its outer boundary $r = 1$. The device used to fix the plate at its center is assumed to have the form of a disk of a very small radius b, compared to that of the central disk [12]. As mentioned in Section 1, we can limit ourselves to vibrations having at least one, or at least two diameters of nodes. This means that the plate fixed at its center can be replaced by a free plate also having at least one, or at least two diameters of nodes ([13], p. 15). Fixing the plate at its center then amounts to assume that for this free plate satisfies

$$\varphi_1(0,\theta) = 0, \tag{6}$$

and

$$\left.\frac{\partial \varphi_1(r,\ \theta)}{\partial r}\right|_{r=0} = 0, \tag{7}$$

Physically, (7) means that the slope of the plate is zero at its center.

For the conditions applying at the outer boundary of the plate, we have that φ_2 must satisfy the following equations related to the local sectional bending moment and shear force [10]

$$\left.\frac{\partial^2 \varphi_2(r,\theta)}{\partial r^2} + \nu_2\left(\frac{\partial \varphi_2(r,\theta)}{\partial r} + \frac{\partial^2 \varphi_2(r,\theta)}{\partial \theta^2}\right)\right|_{r=1} = 0, \tag{8}$$

$$\left.\frac{\partial}{\partial r}\left(\frac{\partial^2 \varphi_2(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_2(r,\theta)}{\partial r}\right) + \frac{\partial^2}{\partial \theta^2}\left((2-\nu_2)\frac{\partial \varphi_2(r,\theta)}{\partial r} - (3-\nu_2)\varphi_2(r,\theta)\right)\right|_{r=1} = 0, \tag{9}$$

We now consider the conditions applying to the functions φ_1 and φ_2 at their junction along the circle $r = a$. The continuity of the solution on this circle first implies that

$$\varphi_1(a,\theta)T_1(t) = \varphi_2(a,\theta)T_2(t), \tag{10}$$

for $0 \leq \theta < 2\pi$ and $t \in \boldsymbol{R}_+$. To reflect the fact that the plate does not undergo radial deformation forming an angle along the circle $r = a$, we have the condition

$$\left.\frac{\partial \varphi_1(r,\theta)}{\partial r}\right|_{r=a} T_1(t) = \left.\frac{\partial \varphi_2(r,\theta)}{\partial r}\right|_{r=a} T_2(t) \tag{11}$$

for $0 \leq \theta < 2\pi$ and $t \in \boldsymbol{R}_+$. Finally, still following from Hamilton's principle and related to the sectional bending moment and shear force applying at $r = a$, we have the following equations for $0 \leq \theta < 2\pi$ and $t \in \boldsymbol{R}_+$

$$\left.\left[\frac{\partial^2 \varphi_1(r,\theta)}{\partial r^2} + \nu_1\left(\frac{1}{a}\frac{\partial \varphi_1(r,\theta)}{\partial r} + \frac{1}{a^2}\frac{\partial^2 \varphi_1(r,\theta)}{\partial \theta^2}\right)\right]\right|_{r=a} T_1(t) = \left.\left[\frac{\partial^2 \varphi_2(r,\theta)}{\partial r^2} + \nu_2\left(\frac{1}{a}\frac{\partial \varphi_2(r,\theta)}{\partial r} + \frac{1}{a^2}\frac{\partial^2 \varphi_2(r,\theta)}{\partial \theta^2}\right)\right]\right|_{r=a} T_2(t) \tag{12}$$

and

$$\left.\left[\frac{\partial}{\partial r}\left(\frac{\partial^2 \varphi_1(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_1(r,\theta)}{\partial r}\right) + \frac{\partial^2}{\partial \theta^2}\left(\frac{2-\nu_1}{a^2}\frac{\partial \varphi_1(r,\theta)}{\partial r} - \frac{3-\nu_1}{a^3}\varphi_1(r,\theta)\right)\right]\right|_{r=a} T_1(t) = \left.\left[\frac{\partial}{\partial r}\left(\frac{\partial^2 \varphi_2(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_2(r,\theta)}{\partial r}\right) + \frac{\partial^2}{\partial \theta^2}\left(\frac{2-\nu_2}{a^2}\frac{\partial \varphi_2(r,\theta)}{\partial r} - \frac{3-\nu_2}{a^3}\varphi_2(r,\theta)\right)\right]\right|_{r=a} T_2(t) \tag{13}$$

3. OPTIMIZATION METHOD

The eigenvalue equation applying to the above problem is derived in Appendix A. It can be formulated as

$$W_1(x)W_4(x) - W_2(x)W_3(x) = 0, \tag{14}$$

where $W_i(x), i = 1, 2, 3, 4,$ are expressions in terms of the Bessel functions J_m and I_m, as well as of the parameters a, ν_j, $j = 1, 2$, and

$$\eta = \frac{c_2}{c_1} = \left(\frac{E_2 h_2^2 \rho_1 (1-\nu_1{}^2)}{E_1 h_1^2 \rho_2 (1-\nu_2{}^2)}\right)^{\frac{1}{4}}, \tag{15}$$

For each set of values for the parameters a, E_j, h_j, ρ_j and ν_j, $j = 1, 2$, Eq. (14) yields an infinite number of roots $x = x_{mn}$, where the indices mn stands for the mode of vibration (m, n) of the plate. The root x_{mn} corresponds to the eigenfrequency f_{mn}= $(x_{mn}c_2)^2/2\pi$. The eigenfrequency corresponding to the fundamental frequency of the plate is chosen to be that whose mode is (m, n) = (2,0), even when the plate has eigenfrequencies smaller than that of the mode (2,0). This makes sense because these frequencies correspond to vibrations having very weak acoustic energy, compared to that of the mode (2,0), and its first higher overtones (see e.g. [1]). Similarly, the roots related to high eigenfrequencies will be neglected, again because of their very weak acoustic energy. By allowing a, E_j, h_j, ρ_j and ν_j, $j = 1, 2$, to take their values within large sets, we can determine the configurations so that the plate vibrates with minimal disharmony, when its two regions have different thicknesses of the same material or are made up of different materials of the same or different thicknesses.

To determine the roots of (14), we apply the secant method starting at $x = 0$ with a step size of 0.001. Therefore, we evaluate the function formed by the left-hand side of (14) at each of these values of x, a solution of this equation being defined as the midpoint of the interval whose extremities give opposite sign values for this function. Like the vibrations of circular membranes, but unlike those produced by a regular string, the vibrations of a circular plate that contribute to a given quasi-harmonic eigenvalue may come from more than one mode of vibration, which are acoustically difficult to distinguish between each other. For this reason, we link to the same harmonic k all roots of (14) such that f_{mn}/f_{20} is in interval $[k - 1/2, k + 1/2]$.

To measure the global deviation of calculated eigenfrequencies from complete harmony, we use the function

$$F(a, h_j, \rho_j, \nu_j\, E_j) = \sum_{k=1}^{5} \left(\frac{\frac{f_{mn}}{f_{20}} - k}{k} \right)^2 , \qquad (16)$$

where $j = 1, 2$ and k represents the overtone rank. Each deviation from harmony is divided by the rank of the corresponding harmonic to express the fact that the energy of a mode generally decreases with its rank. The choice of 5 as upper bound in the summation is somewhat arbitrary but remains consistent with previous studies, higher overtones being greatly damped due to the weakness of their energies (see e.g. [2,15]). Since cymbals might have vibration modes with at least one or two nodal diameters, that is they are of the type (m, n), $m, n \in N$, $m \geq 1$, or $m \geq 2$, we shall consider these two cases separately.

4. TWO MATERIAL CIRCULAR PLATES

To exemplify the above mathematical method, we shall now determine the values of physical parameters of two-region circular plates made of two materials of the same thickness which would vibrate with minimal disharmony. To be able to directly compare our results below with those of Table 1, applying to a single material plate with Poisson's coefficient ν = 0.33 (see e.g. [13], p.11), the two selected materials will have the same Poisson's coefficient.

Table 1. The first five quasi-harmonic normalized angular frequencies ω_j of a circular plate made of a single material with Poisson's coefficient $\nu = 0.33$, for $m \geq 1$. The upper indices (m, n) describes the vibration mode. The sixth column presents the corresponding values of F, the number between parenthesis being this value for $m \geq 2$, which results by eliminating the normalized frequencies associated with $m = 1$ in the preceding columns.

ω_1	ω_2	ω_3	ω_4	ω_5	F
$1.000^{(2,0)}$	$1.525^{(3,0)}$ $(1.975^{(1,1)})$ $2.023^{(4,0)}$	$2.507^{(5,0)}$ $2.588^{(2,1)}$ $2.983^{(6,0)}$ $3.172^{(3,1)}$ $(3.373^{(1,2)})$	$3.735^{(4,1)}$ $4.005^{(2,2)}$ $4.283^{(5,1)}$	$4.609^{(3,2)}$ $(4.753^{(1,3)})$ $4.819^{(6,1)}$ $5.202^{(4,2)}$ $5.399^{(2,3)}$	0.1483 (0.1302)

These materials are aluminum (Al) and copper (Cu). Their other relevant parameters are: ρ_{Al}= 2700 kg/m^3, ρ_{Cu} = 8920 kg/m^3, E_{Al} = 69 GPa and E_{Cu} = 124 GPa. Since $h_1 = h_2$, $\nu_1 = \nu_2$ and the values of ρ_1, ρ_2 and E_1, E_2 are now known, the value of η in (15) is fixed. Consequently, (16) becomes a function F^* of the radius a only, and our optimization problem boils down to determining a that minimizes $F^*(a)$, for $m \geq 2$ and $m \geq 1$, separately. We do this by allowing a to take all values between 0.01 and 0.99 with a step size of 0.01.

If aluminum forms the central disk and copper the ring around it, then η = 0.8588. Applying the solutions of Eq. (14) obtained through the secant numerical method, we find that for $m \geq 2$ the minimum of $F^*(a)$ equals 0.0609 at $a = 0.54$. For $m \geq 1$ this minimum is 0.1293 at $a = 0.2$.

The graphs of $F^*(a)$ for these two cases are in Figure 2. If copper forms the central disk and aluminum the ring around it, then η = 1.524. For $m \geq 2$, the minimum of $F^*(a)$ is 0.1057 at $a = 0.05$. For $m \geq 1$ this minimum is equal to 0.1225 at $a = 0.34$. The graphs describing $F^*(a)$ for these two last cases are shown in Figure 3. Table 2 summarizes the results for these two materials. The minimal value of $F^*(a)$ in the seventh column of Table 2 is for a plate formed by a central disk of aluminum and a ring of copper.

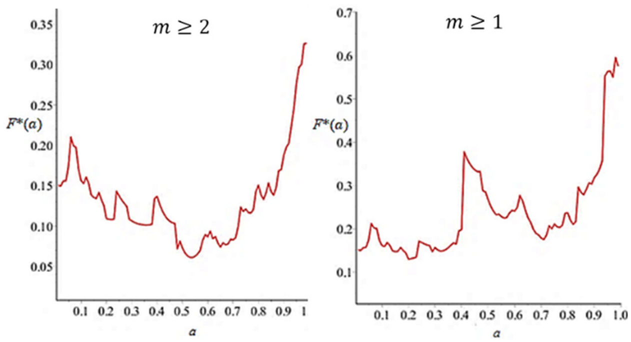

Figure 2. The graphs of $F^*(a)$ for a plate with a central disk of aluminum and a ring of copper for $m \geq 2$ (left) and $m \geq 1$ (right)

This minimum corresponds to a cymbal manufactured so that $m \geq 2$. The radius of its central region is 54% of that of the cymbal. Given that complete harmony, in relation to f_{20}, would be to have $f_{mn}/f_{20} = k$ for the set of f_{mn} corresponding to k = 1, 2, 3, 4, 5, we see that this set of frequencies are close to be harmonic.

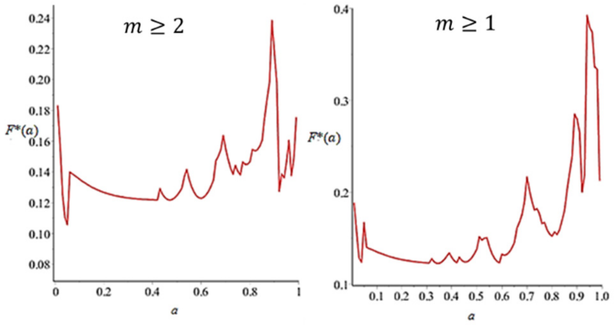

Figure 3. The graphs of $F^*(a)$ for a plate with a central disk of copper and a ring of aluminum for $m \geq 2$ (left) and $m \geq 1$ (right)

For these two materials, the greatest contribution to overall deviation from the first five harmonics comes from the mode (3,0), which contributes to the second harmonic. The percentage of relative deviation of this mode, compared to the second harmonic is always at least 22%. The mode (5,0) is absent from the optimal case where the plate is formed by a disk of aluminum and a ring of copper ring for $m \geq 2$. In all other cases, including single material plates, the mode (5,0) ranks second in terms of the overall deviation of the first five quasi-harmonics.

Table 2. The first five quasi-harmonic normalized angular frequencies ω_j of optimal compound circular plates made of aluminum and copper.

plate	ω_1	ω_2	ω_3	ω_4	ω_5	minimum of $F^*(a)$
disk: Al ring: Cu $m \geq 2$	$1.000^{(2,0)}$	$1.557^{(3,0)}$ $1.815^{(2,1)}$ $2.117^{(3,1)}$		$3.976^{(2,2)}$	$4.932^{(3,2)}$	$F^*(0.54)$ =0.0609
disk: Al ring: Cu $m \geq 1$	$1.000^{(2,0)}$	$1.530^{(3,0)}$ $1.715^{(1,0)}$ $2.041^{(4,0)}$ $2.189^{(2,1)}$	$2.565^{(5,0)}$ $2.580^{(3,1)}$ $2.906^{(4,1)}$ $3.165^{(5,1)}$			$F^*(0.20)$ =0.1293
disk: Cu ring: Al $m \geq 2$	$1.000^{(2,0)}$	$1.524^{(3,0)}$ $2.020^{(4,0)}$	$2.501^{(5,0)}$ $2.970^{(6,0)}$ $3.428^{(7,0)}$	$3.873^{(8,0)}$		$F^*(0.05)$ =0.1057
disk: Cu ring: Al $m \geq 1$	$1.000^{(2,0)}$	$1.512^{(3,0)}$ $1.981^{(4,0)}$ $2.418^{(5,0)}$	$2.830^{(6,0)}$ $3.220^{(7,0)}$	$3.591^{(8,0)}$	$5.049^{(1,0)}$	$F^*(0.34)$ =0.1225

5. CONCLUDING REMARKS

To theoretically determine new configurations of a cymbal that would vibrate with minimal disharmony, we have developed a mathematical method based on the similarity of its vibrations to those of a thin circular plate. This method was applied to a circular plate made of aluminum or copper on a centered disk and a ring around it. The method can be applied directly to other pairs of materials, or to two thicknesses of a single material on a central disk and a ring around it. The mathematical modelling and method of the present paper also extend to the analysis of cymbals composed of more than two materials and/or more than two thicknesses. Finally, note that our results are of a mathematical nature and need to be validated experimentally.

APPENDIX A

In this appendix, we use the method of separation of variables to deduce the eigenvalue equation resulting from (6) - (13). We start by setting

$$\psi_j(r,\theta,t) = R_j(r)\,\Theta_j(\theta)\,T_j(t), \quad j = 1, 2, \qquad \text{(A.1)}$$

where R_j, Θ_j and T_j are functions of r, θ and t, respectively. Substituting (A.1) into (1), one easily shows that

$$R_j(r) = A_{jm_j} J_{m_j}(r\lambda_j) + B_{jm_j} I_{m_j}(r\lambda_j) + C_{jm_j} Y_{m_j}(r\lambda_j) + D_{jm_j} K_{m_j}(r\lambda_j), \qquad \text{(A.2)}$$

where J_k, Y_k and I_k, K_k are the Bessel and modified Bessel functions of order k of the first and second kinds, respectively, and

$$T_j(t) = G_j \sin\left(\lambda_j^2 c_j^2 t + \alpha_j\right), \qquad \text{(A. 3)}$$

$$\Theta_j(\theta) = H_j \sin\left(m_j\theta + \gamma_j\right), \qquad \text{(A. 4)}$$

Here A_{jm_j}, B_{jm_j}, C_{jm_j}, D_{jm_j}, G_j, H_j, α_j and γ_j are still arbitrary constants.

We now consider the substitution of (A.1) into (6) - (13). First, observe that (7) is automatically satisfied when the vibrations have at least one, or at least two, nodal diameters [13, 14]. Putting (A.1) into (6), (10) and (11) gives

$$R_1(0) = 0, \tag{A.5}$$

$$\Theta_1(\theta)R_1(a)T_1(t) = \Theta_2(\theta)R_2(a)T_2(t), \tag{A.6}$$

and

$$\Theta_1(\theta)R_1'(r)T_1(t)|_{r=a} = \Theta_2(\theta)R_2'(r)T_2(t)|_{r=a}, \tag{A.7}$$

But we can also write (A.6) as

$$\frac{T_2(t)}{T_1(t)} = \frac{R_1(a)}{R_2(a)}\frac{\Theta_1(\theta)}{\Theta_2(\theta)}, \tag{A.8}$$

Since the left-hand side of (A.8) is a function of t only and its right-hand side is a function of θ only, we get

$$T_2(t) = \alpha T_1(t), \tag{A.9}$$

and

$$\Theta_2(\theta) = \frac{R_1(a)}{\alpha R_2(a)}\Theta_1(\theta), \tag{A.10}$$

where α is an arbitrary constant. Then, from (4) and (A.9), it follows that

$$\lambda_1^2\, c_1^2 = \lambda_2^2\, c_2^2, \tag{A.11}$$

Therefore, the angular frequency of the plate vibrations can be given by $\omega = \lambda_2^2\, c_2^2$. The functions Θ_j being 2π-periodic, the application of (A.4) to (A.10) leads to $m_1 = m_2 = m \in \boldsymbol{N}$, which implies that m can replace m_j in (A.2) and (A.4). Since $|Y_m(r\lambda_j)| \to \infty$ and $|K_m(r\lambda_j)| \to \infty$ as $r \to 0$, from (A.5) we get $C_{jm} = D_{jm} = 0$ in (A.2).

Moreover, since $c_j > 0$ and only the nonnegative values of λ_j, $j = 1, 2$, are of interest (A.9) implies that $\lambda_1 c_1 = \lambda_2 c_2$. Setting $x = \lambda_2$, it follows that $\lambda_1 = \eta x$ where $\eta = c_2/c_1$ is given by (15). Consequently, for $m \in \boldsymbol{N}$, $m \geq 1$ or $m \geq 2$, we get:

$$R_1(r) = A_{1m}J_m(\eta x r) + B_{1m}I_m(\eta x r), \tag{A.12}$$

$$R_2(r) = A_{2m}J_m(xr) + B_{2m}I_m(xr), \tag{A.13}$$

The eigenvalue equation of this problem will result from the conditions on constants A_{1m}, B_{1m}, A_{2m} and B_{2m} that follow from (6)-(13).

Substituting (A.1) into (12) and (13) gives, respectively,

$$\Theta_1(\theta)T_1(t)\left[R_1''(r) + \frac{\nu_1}{a}\left(R_1'(r) - \frac{m^2}{a}R_1(r)\right)\right]\Bigg|_{r=a} = \Theta_2(\theta)T_2(t)\left[R_2''(r) + \frac{\nu_2}{a}\left(R_2'(r) - \frac{m^2}{a}R_2(r)\right)\right]\Bigg|_{r=a}, \tag{A.14}$$

and

$$\Theta_1(\theta)T_1(t)\cdot\left[\left(R_1'''(r) + \frac{R_1''(r)}{r} - \frac{R_1'(r)}{r^2}\right) - \frac{m^2}{a^2}\left((2-\nu_1)R_1'(r) - \frac{(3-\nu_1)}{a}R_1(r)\right)\right]\Bigg|_{r=a} = \Theta_2(\theta)T_2(t)\cdot\left[\left(R_2'''(r) + \frac{R_2''(r)}{r} - \frac{R_2'(r)}{r^2}\right) - \frac{m^2}{a^2}\left((2-\nu_2)R_2'(r) - \frac{(3-\nu_2)}{a}R_2(r)\right)\right]\Bigg|_{r=a}, \tag{A.15}$$

The substitution of (A.1) into (8) and (9) also leads to:

$$R_2''(r) + \nu_2\left(R_2'(r) - m^2 R_2(r)\right)\big|_{r=1} = 0, \tag{A.16}$$

and

$$R_2'''(r) + R_2''(r) - R_2'(r) - m^2\left((2-\nu_2)R_2'(r) - (3-\nu_2)R_2(r)\right)\Big|_{r=1} = 0, \tag{A.17}$$

From (A.6) and (A.14), we deduce:

$$R_2(a)\left[R_1''(r) + \frac{\nu_1}{a}R_1'(r) - \frac{m^2\nu_1}{a^2}R_1(r)\right]\Bigg|_{r=a} = R_1(a)\left[R_2''(r) + \frac{\nu_2}{a}R_2'(r) - \frac{m^2\nu_2}{a^2}R_2(r)\right]\Bigg|_{r=a}, \tag{A.18}$$

Similarly, from (A.6) and (A.15), we get:

$$R_2(a)\left[R_1'''(r) + \frac{R_1''(r)}{a} - \frac{1+m^2(2-\nu_1)}{a^2}R_1'(r) + \frac{m^2(3-\nu_1)}{a^3}R_1(a)\right]\Bigg|_{r=a} = R_1(a)\left[R_2'''(r) + \frac{R_2''(r)}{a} - \frac{1+m^2(2-\nu_2)}{a^2}R_2'(r) + \frac{m^2(3-\nu_2)}{a^3}R_2(r)\right]\Bigg|_{r=a}, \tag{A.19}$$

Substituting (A.12) and (A.13) into (A.18) gives:

$$[A_{2m}X_1(x)+B_{2m}X_2(x)][A_{1m}X_3(x)+B_{1m}X_4(x)] = [A_{1m}Y_1(x)+B_{1m}Y_2(x)][A_{2m}Y_3(x)+B_{2m}Y_4(x)], \quad (A.20)$$

where:

$X_1(x)=J_m(xa), X_2(x)=I_m(xa),$
$Y_1(x)=J_m(\eta xa), Y_2(x)=I_m(\eta xa),$
$X_3(x)=\eta^2x^2J_m''(\eta xa)+\frac{\nu_1\eta x}{a}J_m'(\eta xa)-\frac{m^2\nu_1}{a^2}J_m(\eta xa),$
$X_4(x)=\eta^2x^2I_m''(\eta xa)+\frac{\nu_1\eta x}{a}I_m'(\eta xa)-\frac{m^2\nu_1}{a^2}I_m(\eta xa),$
$Y_3(x)=x^2J_m''(xa)+\frac{\nu_2 x}{a}J_m'(xa)-\frac{m^2\nu_2}{a^2}J_m(xa),$
$Y_4(x)=x^2I_m''(xa)+\frac{\nu_2 x}{a}I_m'(xa)-\frac{m^2\nu_2}{a^2}I_m(xa).$

Similarly, by substituting (A.12) and (A.13) into (A.19), we get:

$$[A_{2m}X_1(x)+B_{2m}X_2(x)][A_{1m}Z_1(x)+B_{1m}Z_2(x)] = [A_{1m}Y_1(x)+B_{1m}Y_2(x)][A_{2m}Z_3(x)+B_{2m}Z_4(x)], \quad (A.21)$$

where

$Z_1(x)=\eta^3x^3J_m'''(\eta xa)+\frac{\eta^2x^2}{a}J_m''(\eta xa) -\frac{(1+m^2(2-\nu_1))\eta x}{a^2}J_m'(\eta xa)+\frac{m^2(3-\nu_1)}{a^3}J_m(\eta xa)$
$Z_2(x)=\eta^3x^3I_m'''(\eta xa)+\frac{\eta^2x^2}{a}I_m''(\eta xa) -\frac{(1+m^2(2-\nu_1))\eta x}{a^2}I_m'(\eta xa)+\frac{m^2(3-\nu_1)}{a^3}I_m(\eta xa)$
$Z_3(x)=x^3J_m'''(xa)+\frac{x^2}{a}J_m''(xa) -\frac{(1+m^2(2-\nu_2))x}{a^2}J_m'(xa)+\frac{m^2(3-\nu_2)}{a^3}J_m(xa)$
$Z_4(x)=x^3I_m'''(ax)+\frac{x^2}{a}I_m''(ax) -\frac{(1+m^2(2-\nu_2))x}{a^2}I_m'(ax)+\frac{m^2(3-\nu_2)}{a^3}I_m(xa).$

The substitution of (A.13) into (A.16) gives:

$$A_{2m}S_1(x)+B_{2m}S_2(x)=0, \quad (A.22)$$

where

$S_1(x)=x^2J_m''(x)+\nu_2xJ_m'(x)-\nu_2m^2J_m(x),$
$S_2(x)=x^2I_m''(x)+\nu_2xI_m'(x)-\nu_2m^2I_m(x)$

Also, by substituting (A.13) into (A.17), we get:

$$A_{2m}T_1(x)+B_{2m}T_2(x)=0, \quad (A.23)$$

where

$T_1(x)=x^3J_m'''(x)+x^2J_m''(x) -(1+m^2(2-\nu_2))xJ_m'(x)+m^2(3-\nu_2)J_m(x)$
$T_2(x)=x^3I_m'''(x)+x^2I_m''(x) -(1+m^2(2-\nu_2))xI_m'(x)+m^2(3-\nu_2)I_m(x).$

Each of the coefficients A_{1m}, B_{1m}, A_{2m} and B_{2m}, $m\in N$, for $m\geq 1$, or $m\geq 2$, can be nonzero in any solution of the problem formed by (1) and (6)-(13), to which adds initial conditions on the position and speed of the plate. When such a plate vibrates, it is indeed expected that both its central disk and the ring around this disc vibrate.

Therefore, the homogeneous linear system of equations formed by (A.22) and (A.23), for the unknown constants A_{2m} and B_{2m}, will have a specific non-trivial solution, implying that the determinant of the matrix of its coefficients equals zero, that is

$$S_1(x)T_2(x)-S_2(x)T_1(x)=0, \quad (A.24)$$

Hence, the eigenvalues of the present problem must be in the set of solutions of (A.24). Taking as realized the latter condition, we get that at least one coefficient among A_{2m} and B_{2m} is nonzero.

Suppose that A_{2m} is non-zero (a similar reasoning applies when one assumes that B_{2m} is non-zero), and divide (A.20) by A_{2m}. Using (A.22), we obtain:

$$A_{1m}[X_3(x)(X_1(x)S_2(x)-X_2(x)S_1(x))Y_1(x)(Y_3(x)S_2(x)-Y_4(x)S_1(x))] + B_{1m}[X_4(x)(X_1(x)S_2(x)-X_2(x)S_1(x))Y_2(x)(Y_3(x)S_2(x)-Y_4(x)S_1(x))]=0, \quad (A.25)$$

Similarly, dividing (A.21) by A_{2m} and using (A.23) we get:

A_1m[Z_1(x)(X_1(x)T_2(x)
− X_2(x)T_1(x))Y_1(x)(Z_3(x)T_2(x) Z_4 (x)T_1 (x))]
+ B_1m[Z_2(x)(X_1(x)T_2(x)
− X_2(x)T_1(x))Y_2(x)(Z_3(x)T_2(x) − Z_4(x)T_1(x))]
= 0, (A.26)

Seen as a system of linear equations for the unknowns A_{1m} and B_{1m}, the equations (A.25) and (A.26) become:

$$A_{1m}W_1(x)+B_{1m}W_2(x)=0,$$
$$A_{1m}W_3(x)+B_{1m}W_4(x)=0, \quad (A.27)$$

where

$W_1(x)=X_3(x)(X_1(x)S_2(x)-X_2(x)S_1(x)) -Y_1(x)(Y_3(x)S_2(x)-Y_4(x)S_1(x))$
$W_2(x)=X_4(x)(X_1(x)S_2(x)-X_2(x)S_1(x)) -Y_2(x)(Y_3(x)S_2(x)-Y_4(x)S_1(x))$
$W_3(x)=Z_1(x)(X_1(x)T_2(x)-X_2(x)T_1(x)) -Y_1(x)(Z_3(x)T_2(x)-Z_4(x)T_1(x))$
$W_4(x)=Z_2(x)(X_1(x)T_2(x)-X_2(x)T_1(x)) -Y_2(x)(Z_3(x)T_2(x)-Z_4(x)T_1(x))$

This system of equations will have a non-trivial solution for A_{1m} and B_{1m} if and only if the determinant of its matrix of coefficients equals zero, which is (14) of main text.

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