Free Vibration of Initially Deflected Axially Functionally Graded Non-Uniform Timoshenko Beams on Elastic Foundation

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Abstract: - The present paper studies the effect of elastic foundation on free vibration of initially deflected non-uniform axially functionally graded (AFG) thick beam on elastic foundation on the basis of Timoshenko beam theory. The elastic foundation is idealized as a set of parallel linear springs. Formulation is carried out through displacement based energy principle. First, the static problem is solved to find out the unknown displacement field by using minimum total potential energy principle. Solution methodology involves an iterative technique known as direct substitution with relaxation scheme. Secondly, subsequent dynamic problem is set up as an eigenvalue problem on the basis of the known displacement field. The governing set of equations in dynamic problem is obtained by using Hamilton's principle and solved with the help of Matlab's intrinsic solver. The results of the present method are validated with previously published articles. Frequency vs. displacement curve corresponding to different combination of system parameters are presented in non-dimensional plane and are capable of serving as benchmark results.

Keywords: - AFG beam, Timoshenko beam, non-uniform beam, geometric non-linearity, elastic foundation, energy principle.

1. INTRODUCTION

Non-uniform and inhomogeneous beams are extensively used in several engineering applications. Such structural elements find useful utilization across various disciplines and domains, such as, civil, mechanical, marine, aerospace and biomedical sectors. As a result, relevant static, dynamic and buckling issues have been quite frequently and extensively highlighted by researchers and engineers. Especially now a day, beams made of functionally graded materials (FGM) are of great interest due to their excellent thermo-mechanical properties. They can also exhibit some special characteristics, such as thermal resistance, high stiffness and low delamination difficulties. FGMs are obtained by varying the volume fraction of the constituent materials (usually, ceramic and metal) continuously and functionally along one or more of the spatial directions. Thus, material properties of the inhomogeneous mixture show gradation along those directions. Considering the variation of properties along with non-uniform cross-section, structural analysis becomes more complex. Especially, for axially functionally graded (AFG) beams with non-uniform section, the governing equations involve several variable coefficients adding to the level of complexity.

It has to be mentioned here that depending on the direction of gradation, three different types of functionally graded beams can be considered, namely, transversely functionally graded (TFG), axially functionally graded (AFG) and bi-directional functionally graded (BFG). In case of transversely graded beam, material property variation is along the thickness direction, while, for axially graded beams, material property varies along the length of the beam. In BFG, the material properties vary continuously along thickness and longitudinal directions simultaneously. A thorough review of existing literature on functionally graded beams indicates that there are a huge number of research papers devoted towards analysis of transversely graded beams. Comparatively the domain of axially graded beams is more recent and hence, the focus of the present work is on AFG thick beams.

There are several articles devoted to explore static, dynamic and buckling behaviour of axially functionally graded (AFG) non-uniform thick beams based on Timoshenko beam theory. Due to the presence of variable coefficients and non-linear equations, exact solutions of the governing equations are generally unavailable. Therefore, several numerical methods have been used to obtain solutions to AFG beam problems. In the following paragraph, selected research works are highlighted in order to set the backdrop for the present analysis.

El-Ashmawy et al [1] proposed a generalized non-conventional finite element (FE) model for beam following Timoshenko beam theory and performed static and dynamic analysis for AFG Beam. Zhao et al [2] introduced a new approach based on Chebyshev polynomials theory to study free vibrational behaviours of AFG Timoshenko beams with tapered cross-sections. Lagrange's equation was applied to obtain the discrete governing equation. Sari et al [3] developed a model for beam structure based on Timoshenko beam theory for AFG non-uniform nanobeams with Eringen's nonlocal residuals and studied the effects of the residuals on the natural frequencies and mode shapes. Kiani [4] studied the transverse vibration of AFG tapered nanobeams in a longitudinal temperature gradient. Based on the hypotheses of the Rayleigh, higher-order, and Timoshenko beam theory, the equations of motion of the nanobeam were displayed using surface elasticity theory of Gurtin–Murdoch. Huang et al [5] investigated the buckling behaviour of AFG and tapered Timoshenko beams. Auxiliary function and power series were employed to covert the coupled equations into a system of linear algebraic equations to study the buckling characteristics. Chen et al [6] conducted free vibration analysis of AFG nanocantilevers along with nanoparticle. Timoshenko beam theory incorporating surface effects was utilized to study the vibrational behaviours. Sarkar and Ganguli [7] studied the free vibration of AFG Timoshenko beams, with uniform cross-section, having fixed–fixed boundary condition. Shafiei et al [8] presented a study on the small scale effect on vibration of a rotary AFG microbeam on the basis of modified couple stress and Timoshenko beam theories. Hamilton's principle was employed to derive the equations and the generalized differential quadrature method was utilized to solve it. Huang et al [9] studied free vibration a spinning AFG Timoshenko beam by employing spectral-Tchebychev method. Wang and Wu [10] investigated the dynamic response of an AFG beam under thermal environment and subjected to a moving harmonic load on the basis of classical beam theory and Timoshenko beam theory. Lagrange method was employed to derive the equations. Calim [11] performed transient analysis of AFG Timoshenko beams with non-uniform cross-section. Complementary functions method was utilized to solve the differential equations in Laplace domain and modified Durbin's algorithm was applied to transform the results into the time domain. Shahba et al [12] conducted the free vibration and stability analysis of AFG non-uniform Timoshenko beams through finite element. Yong Huang et al [13] presented a new approach for investigating the vibration behaviours of AFG Timoshenko non-uniform beams introducing an auxiliary function which convert coupled governing equations with variable coefficients to a single governing equation. Rajasekaran [14] studied free transverse vibration of rotating AFG Timoshenko tapered beams using Differential Transformation method and differential quadrature element method of lowest order.

Study of behaviour of beams on elastic foundation is an interesting domain of research as several critical engineering structures can be idealized as beams on foundation. Issues related to such structures are taken up for investigation because they belong to a class of frequently used structural elements which generally serve as the key load-bearing components, like rail track, rigid pavements, bridge decks, mat and raft foundations etc. Literature review reveals that there exist a number of papers related to FG Timoshenko beam on elastic foundation.

Yas and Samadi [15] studied free vibration and buckling behaviour of nanocomposite Timoshenko beams reinforced by single-walled carbon nanotubes on an elastic foundation. The governing equations were derived through employing Hamilton's principle and solved by utilizing the generalized differential quadrature method. Komijani et al [16] investigated buckling and post-buckling behaviour and vibrations of FG Timoshenko beams rested on nonlinear elastic foundation and subjected to in-plane thermal loads. The von Kármán nonlinearity and modified couple stress theory were employed to derive the governing nonlinear equations. Generalized differential quadrature method was used to discretize the motion equation and Newton's method was used to solve the nonlinear algebraic equations. Mohanty et al [17] investigated the dynamic stability of FG Timoshenko beam and FG sandwich (FGSW) beam on Winkler elastic foundation through FE method. Tossapanon and Wattanasakulpong [18] utilized Chebyshev collocation method to solve buckling and vibration problems of FG sandwich beams resting on two-parameter elastic foundation on the basis of Timoshenko beam theory in order to incorporate the significant effects of shear deformation and rotary inertia. Deng et al [19] proposed an exact solution of double- FG Timoshenko beams on Winkler-Pasternk elastic foundation. Hamilton's principle was used to derive the differential equations of motion. Arefi and

Zenkour [20] studied wave propagation analysis of a FG nanobeam made of magneto-electro-elastic materials and rested on Visco-Pasternak foundation using Timoshenko beam model. Surface elasticity was applied for modelling the behaviour of nanobeam. Hamilton principle was used to derive the equations of motion. Arefi and Zenkour [21] studied wave propagation analysis for a FG nanobeam with rectangular cross-section on visco-Pasternak's foundation using Timoshenko's beam model and nonlocal elasticity theory. The equations of motion were derived using Hamilton's principle. Yan et al [22] studied the dynamic response of FG beams with an open edge crack supported on elastic foundation and subjected to a moving transverse load. Timoshenko beam theory was used in theoretical formulations to account for the transverse shear deformation. Esfahani et al [23] examined thermal buckling and post-buckling analysis of FGM Timoshenko beams on a non-linear elastic foundation. Timoshenko beam theory and von-Karman's strain–displacement relations were employed to obtain the non-linear equations. Generalized Differential Quadrature Method was applied to solve the non-linear equations in space domain.

The research papers described in the above paragraph dealt with transversely graded Timoshenko beams on elastic foundation. On the other hand, AFG Timoshenko beam on elastic foundation is a more recent domain of research and till now, only a few articles are available in literature in this field. Calim [24] analysed free and forced vibrations of AFG Timoshenko beams on two-parameter viscoelastic foundation. Complementary functions method was utilized to solve the differential equations in Laplace domain and modified Durbin's algorithm was applied to transform the results into the time domain. However, there are a few research works relating to elastic foundation supported AFG thin beams, where Euler-Bernoulli theory is utilised for mathematical formulation of the problem. Huang and Luo [25] presented a new and simple method to calculate the critical buckling loads of beams with axial inhomogeneity on elastic foundation, whereas, Lohar et al [26,27] investigated the dynamic behaviour of AFG beam on elastic foundation.

Considering the relative scarcity of research work on AFG Timoshenko beams on elastic foundation, the present study is focussed in that particular domain. It has been already mentioned that a beam on foundation can serve as an idealised model of critical load bearing components. Hence, there is a need to analyse the system behaviour under loading. Majority of free vibration studies concentrate on determination of vibration frequencies for different modes as well as mode shapes. But such a study is obviously carried out under no load condition. However, in the present scenario, a transversely loaded non-uniform AFG Timoshenko beam on elastic foundation is considered. The problem can be distinctly broken up into two parts. At first the nonlinear static problem incorporating large deflection under transverse loading is taken up. Then the subsequent dynamic problem is formulated employing Hamilton's principle and utilizing the previously solved static displacement field. Vibration frequencies and mode shapes are determined corresponding to the deflected configuration of the beam and results are provided as backbone curves and mode shape plots. The effects of the elastic foundation, axial material gradation, non-uniformity of beam, end conditions etc. are studied.

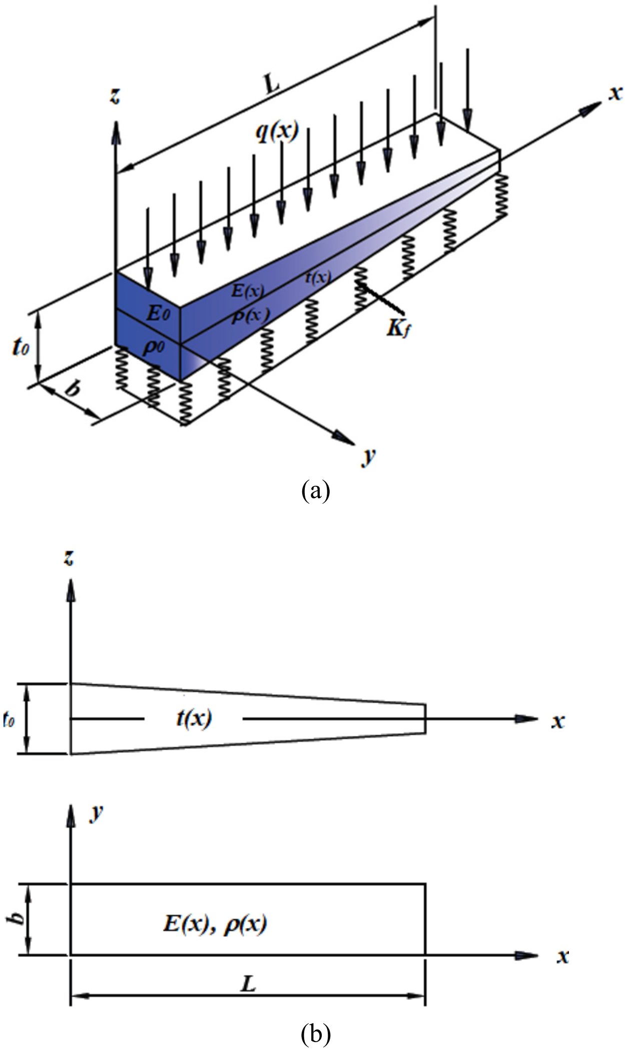


Figure 1. (a) AFG Beam on Elastic Foundation, (b) Front and Top view of the Taper Beam

2. MATHEMATICAL FORMULATION

A non-uniform AFG beam of length L, width b and variable thickness $t(x)$, subjected to uniformly distributed transverse loading is considered as shown in Figure 1. The variation of thickness of the beam, *t(x),* is along the axial direction [Figure 1(b)]. Similarly, the gradation of material properties i.e. modulus of elasticity, *E(x),* and the mass density, *ρ(x),* of the beam is considered along the longitudinal axis (x-axis) of the beam [Figure 1(b)]. The thickness of the beam at the origin is identified as the root thickness and represented by t_0. Similarly, at that same cross-section of the beam the elastic modulus and density are denoted as E_0 and ρ_0, respectively. Although the present study only considers tapering of thickness, it is possible to incorporate variation of width along the longitudinal axis into the formulation without much difficulty. In the present formulation, the beam is considered to be supported on elastic foundation. The foundation behaviour is assumed to be linear such that the reaction forces applied by the foundation on the beam are proportional to the deflection of the beam. To model this behaviour effectively and to handle beam-foundation interaction, it is idealized as a series of linear springs attached to the bottom surface of the beam [Figure 1(a)]. The foundation stiffness is quantified by the stiffness coefficient of the linear spring and it is denoted by K_f.

Mathematical formulation of the problem is performed in such a way that it can be broken down into two distinct but interlinked problems - first, a static analysis and second, a free vibration analysis based on the static solution [28,29]. The static analysis deals with the system under uniform transverse loading that imposes transverse deflection on the beam. This problem takes into account geometrically nonlinear and hence addresses the large deflection effect. Nonlinear strain-displacement relations are considered in order to incorporate geometric nonlinearity. Once the solution to the static problem is obtained pre-loaded static configuration of the beam under application of uniformly distributed load becomes known. Now, the system is assumed to execute a small amplitude vibration about the deflected configuration. Hence, the free vibration problem is formulated with the objective of finding out the loaded natural frequencies of the deformed system. The effect of the statically inflicted large deflection is incorporated into the dynamic system. Both the static and dynamic analyses are formulated based on variational form of appropriate energy principles. Also, shear deformation and rotary inertia are incorporated by following Timoshenko beam theory.

In the present analysis, three different types of flexural boundary conditions are considered. These are chosen from combinations of clamped (C) and simply supported (S) classical end supports. Hence, CC, CS and SS are the selected boundary conditions which are considered here. However, the formulation is flexible enough to accommodate free (F) end conditions in combination with the other two with minimum modifications. The notations used to denote the boundary conditions are two letters representing the individual end conditions at $x = 0$ and $x = L$, sequentially. The in-plane boundary conditions are assumed to be immovable, i.e., zero displacements along x-axis is imposed at the two extremities of the beam.

2.1. Static analysis

The static analysis is based on the minimum total potential energy principle [30], which states that the total potential energy of the system is minimized by a kinematically admissible displacement field (out of all the kinematically admissible displacement fields) corresponding to the stable equilibrium. The principle is mathematically expressed as,

$$\delta\left(U+V\right)=0 \tag{1}$$

Here, δ designates variational operator, U is the total strain energy stored in the system (which includes the beam as well as the foundation) due to external loadings and V is the work done by the external loading. Total strain energy in the system is made up of strain energy stored in the beam and the foundation. Moreover, strain energy of the beam can again be split into two components, namely, strain energy due to axial strain (U_a) and strain energy due to shear strain (U_s). This implies that the total strain energy can be expressed as the summation of three distinct parts, as follows: $U = U_a + U_s + U_f$. Here, U_f is the strain energy stored in the system due to deformation of the elastic foundation.

The present formulation is a semi-analytical displacement based approximate method, which indicates that in the expression of strain energies, the strains (axial as well as shear) are to be replaced by appropriate displacement relations. In order to express the strain energies in terms of displacement fields, following strain-displacement expressions are invoked [30] for axial strain (ε_a) and shear strain (ε_s).

$$\varepsilon_a=\frac{1}{2}\left(\frac{dw}{dx}\right)^2+\frac{du}{dx}-z\frac{d\psi}{dx} \tag{2}$$

$$\varepsilon_s=\frac{1}{2}\left(\frac{dw}{dx}-\psi\right) \tag{3}$$

It is important to note that Equation (2) is nonlinear in nature and on neglecting the third term of the expression it resembles von Karman type nonlinear strain-displacement relation. The displacement fields associated with the above expressions are as follow: transverse displacement field (w), in-plane displacement field (u) and rotational field (ψ) of beam section due to bending. All these displacement fields are dependent on the axial coordinate and they are defined at the mid-plane of the beam.

Substituting the strain-displacement expressions, the strain energies, U_a, U_s and U_f, can be expressed as follows,

$$U_a = \frac{b}{2}\int_0^L\int_{-\frac{t}{2}}^{\frac{t}{2}} \left\{ \begin{array}{l} \frac{1}{4}\left(\frac{dw}{dx}\right)^4 + \left(\frac{du}{dx}\right)^2 \\ +z^2\left(\frac{d\psi}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 \frac{du}{dx} \\ -z\left(\frac{dw}{dx}\right)^2 \frac{d\psi}{dx} - 2z\frac{du}{dx}\frac{d\psi}{dx} \end{array} \right\} E(x)\,dxdz \quad (4)$$

$$U_s = \frac{bk_{sh}}{2}\int_0^L\int_{-\frac{t}{2}}^{\frac{t}{2}} \left\{ \left(\frac{dw}{dx}\right)^2 - 2\frac{dw}{dx}\psi + \psi^2 \right\} G(x)\,dxdz \quad (5)$$

$$U_f = \frac{1}{2}\int_0^L K_f w^2 dx \quad (6)$$

k_{sh} is termed as shear correction factor. It is well known that for rectangular cross-section the numerical value of k_{sh} is taken as 5/6. Here, $G(x)$ is shear modulus which is expressed by, $G(x) = E(x)/2(1+\mu)$.

In order to implement the minimum total potential energy principle, expression for potential for external work need to be obtained and it is provided corresponding to uniformly distributed load $q(x)$ in Equation (7). It is noteworthy that in the present study, only uniformly distributed load has been taken into account. However, this is not a limitation on the present methodology. Any other form of transverse loading (such as concentrated load, triangular load, hat load etc.) can be accommodated in the analysis, as long as it can be expressed mathematically in terms of analytical or numerical functions.

$$V = \int_0^L qwdw \quad (7)$$

As the present method is an approximate one, the displacement fields need to be approximated through finite linear combinations of kinematically admissible functions and unknown coefficients as shown.

$$w = \sum_{i=1}^{nw} d_i \phi_i(x) \quad (8a)$$

$$u = \sum_{i=1}^{nu} d_{nw+i}\alpha_i(x) \quad (8b)$$

$$\psi = \sum_{i=1}^{nsi} d_{nw+nu+i}\beta_i(x) \quad (8c)$$

In the above expressions, φ_i, α_i and β_i represent the set of orthogonal admissible functions corresponding to the displacement fields w, u and ψ, respectively, whereas, d_i denotes the set of unknown coefficients. The number of functions in these sets are taken as nw, nu and nsi (corresponding to w, u and ψ) and from the point of view of the numerical scheme for solution, $nw = nu = nsi$. The choice of the admissible functions are solely based on the boundary conditions of the beam. In fact, the first function within the set (φ_1, α_1 and β_1) is known as the start function and it is selected carefully satisfying the requisite boundary conditions, i.e., flexural, in-plane and end rotation conditions. It should also be pointed out that these functions need to be continuous and differentiable within the domain. The selected start functions for each of the displacement fields are given in Table 1 for all the three boundary conditions under consideration. Consequently, the higher order functions (upto $nw/nu/nsi$) are generated following a numerical implementation of Gram–Schmidt orthogonalization scheme [31]. Once the approximate displacement fields are substituted into the appropriate energy functionals, the problem reduces to finding out the unknown coefficients (d_i).

Table 1. List of start functions for the displacement fields.

Displace-ment field	*Boundary*	*Conditions*	*Function*
w	CC	$w(0)=0$, $w(L)=0$	$\phi_1=(x/L)\{1-(x/L)\}$
	CS		
	SS		
u	CC	$u(0)=0$, $u(L)=0$	$\alpha_1=(x/L)\{1-(x/L)\}$
	CS		
	SS		
ψ	CC	$\psi(0)=0$, $\psi(L)=0$	$\beta_1=\sin(\pi x/L)$
	CS	$\psi(0)=0$, $\psi(l)\neq 0$	$\beta_1=\sin(\pi x/2L)$
	SS	$\psi(0)\neq 0$, $\psi(L)\neq 0$	$\beta_1=\cos(\pi x/L)$

Substituting the expression of the energy functionals, given by Equations (4), (5), (6) and (7) into Equation (1) and considering the assumed displacement fields, given by Equation (8), the governing set of equations are obtained in matrix form, as follows,

$$[K]\{d\} = \{f\} \quad (9)$$

where, $[K]$ are the stiffness matrix and $\{f\}$ are the load vector. The dimensions of these each matrix will be ($nu + nw + nsi$). The form and elements of $[K]$ and $\{f\}$ are furnished below,

$$\left[K_{ij}\right]_{\substack{i=1,nw\\ j=1,nw}}=\frac{b}{2}\int_0^L\left(\sum_{k=1}^{nw}d_k\frac{d\phi_k}{dx}\right)^2\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}t(x)E(x)dx$$

$$+b\int_0^L\left(\sum_{k=nw+1}^{nw+nu}d_k\frac{d\alpha_{k-nw}}{dx}\right)\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}t(x)E(x)dx$$

$$+k_{sh}b\int_0^L\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}t(x)G(x)dx$$

$$\left[K_{ij}\right]_{\substack{i=1,nw\\ j=nw+1,nw+nu}}=0$$

$$\left[K_{ij}\right]_{\substack{i=1,nw\\ j=nw+nu+1,nw+nu+nsi}}=$$

$$-k_{sh}b\int_0^L\frac{d\phi_i}{dx}\beta_{j-nw-nu}t(x)G(x)dx$$

$$\left[K_{ij}\right]_{\substack{i=nw+1,nw+nu\\ j=1,nw}}$$

$$=\frac{b}{2}\int_0^L\left(\sum_{k=1}^{nw}d_k\frac{d\phi_k}{dx}\right)\frac{d\alpha_{i-nw}}{dx}\frac{d\phi_j}{dx}t(x)E(x)dx$$

$$\left[K_{ij}\right]_{\substack{i=nw+1,nw+nu\\ j=nw+1,nw+nu}}=b\int_0^L\frac{d\alpha_{i-nw}}{dx}\frac{d\alpha_{j-nw}}{dx}t(x)E(x)dx$$

$$\left[K_{ij}\right]_{\substack{i=nw+1,nw+nu\\ j=nw+nu+1,nw+nu+nsi}}=0$$

$$\left[K_{ij}\right]_{\substack{i=nw+nu+1,nw+nu+nsi\\ j=1,nw}}=-k_{sh}b\int_0^L\beta_{i-nw-nu}\frac{d\phi_j}{dx}t(x)G(x)dx$$

$$\left[K_{ij}\right]_{\substack{i=nw+nu+1,nw+nu+nsi\\ j=nw+1,nw+nu}}=0$$

$$\left[K_{ij}\right]_{\substack{i=nw+nu+1,nw+nu+nsi\\ j=nw+nu+1,nw+nu+nsi}}=$$

$$\frac{b}{12}\int_0^L\frac{d\beta_{i-nw-nu}}{dx}\frac{d\beta_{j-nw-nu}}{dx}t^3(x)E(x)dx$$

$$+k_{sh}b\int_0^L\beta_{i-nw-nu}\beta_{j-nw-nu}t(x)G(x)dx$$

The elements of the load vector $\{f_i\}$ are,

$$\{f_i\}_{i=1,nw}=q\int_0^L\phi_i dx$$

$$\{f_i\}_{i=nw+1,nw+nu}=0$$

$$\{f_i\}_{i=nw+nu+1,nw+nu+nsi}=0$$

A glance at the elements of the stiffness matrix reveals that the unknown coefficients (d_i) appear in certain terms. It means that stiffness matrix is a function of the undetermined parameters by virtue of considering nonlinear strain-displacement relations and consequently large deformation. Hence, the set of governing equations represented by Equation (9) is nonlinear in nature. A direct solution involving inversion of the stiffness matrix, followed by pre-multiplication with the load vector is not an option. Approximate solution for the unknown coefficients (d_i) from the set of nonlinear equations is obtained following direct substitution method with successive relaxation. This is a numerical iterative technique dependent on an initial guess.

At the initial stage, certain numerical parameters (such as number of Gauss points, number of functions, tolerance on error limit, relaxation parameter etc.) essential to the iterative scheme are fixed and the unknown coefficients are assigned with zero value as an initial guess. With this assumption the stiffness matrix is completely known. In fact, it reduces to the linear form. On the basis of this matrix a new set of unknown coefficients are evaluated through the following expression: $\{d\} = [K]^{-1}\{f\}$. The new set is compared against the previous one to determine the error and if it is above the pre-defined allowable error limit, modification of the evaluated set of unknown coefficients is performed with a relaxation parameter. The process is repeated till the error value falls below the allowable limit and convergence is achieved. At the end of the static analysis, the displacement fields corresponding to transverse load is known and the deflected configuration of the transversely loaded beam is found out. It also implies that the deformed system stiffness is known and can be used in the following free vibration analysis.

2.2. Dynamic analysis

Starting point for the dynamic analysis is Hamilton's principle [30], which is utilized to derive the governing sets equations,

$$\delta\left(\int_{\tau_1}^{\tau_2}(T-U-V)d\tau\right)=0 \tag{10}$$

Here, T is the kinetic energy of the beam due to external excitation, U refers to the system strain energy with respect to deflected configuration and τ is the time. The expression of U remains unchanged from that described by Equations (4) – (6) in static analysis. Kinetic energy of the present system is expressed through the expression shown in Equation (11). It needs to be mentioned here that the present free vibration study is performed on a pre-stressed beam, whose static solution has already been obtained in the previous step. So, the potential energy of the external load (V) can be set to zero. Here, $\rho(x)$ is the mass density of the beam. The

$$T=\frac{b}{2}\int_0^L\int_{-\frac{t}{2}}^{\frac{t}{2}}\left\{\left(\frac{dw}{dt}\right)^2+\left(\frac{du}{dt}\right)^2+z^2\left(\frac{d\psi}{dt}\right)^2\right\}\rho(x)dxdz \tag{11}$$

displacement fields associated with the Expression (11) are dynamic in nature with a time varying component. So, these fields contain a spatial and a temporal part and it is assumed that these two parts are completely separable. The assumed dynamic displacement fields are expressed as follows,

$$w(x,\tau)=\sum_{i=1}^{nw}c_i\phi_i\left(x\right)e^{j\omega\tau} \quad (12a)$$

$$u(x,\tau)=\sum_{i=1}^{nu}c_{nw+i}\alpha_i\left(x\right)e^{j\omega\tau} \quad (12b)$$

$$\psi(x,\tau)=\sum_{i=1}^{nsi}c_{nw+nu+i}\beta_i\left(x\right)e^{j\omega\tau} \quad (12c)$$

Here, the spatial part apparently seems same as that considered in the static analysis and c_is are a set of unknown coefficients different from the unknown parameters defined in the static analysis. However, φ_i, α_i and β_i are same as the static problem. ω denotes the natural frequency of the vibratory system and $j=\sqrt{-1}$. Substituting the expressions of strain energy (Equations (4), (5) and (6)), kinetic energy (Equation (11)) into dynamic displacements (Equation (12)), the governing equation is obtained as follows:

$$\left[K\right]\left\{c\right\}-\omega^2\left[M\right]\left\{c\right\}=0 \quad (13)$$

here, [K] is the stiffness matrix of the system at deflected configuration and [M] is mass matrix, respectively. The elements of the mass matrix $[M_{ij}]$ are,

$$\left[M_{ij}\right]_{\substack{i=1,nw\\ j=1,nw}}=b\int_0^L\phi_i\phi_j t\left(x\right)\rho\left(x\right)dx$$

$$\left[M_{ij}\right]_{\substack{i=nw+1,nw+nu\\ j=nw+1,nw+nu}}=b\int_0^L\alpha_{i-nw}\alpha_{j-nw}t\left(x\right)\rho\left(x\right)dx$$

$$\left[M_{ij}\right]_{\substack{i=nw+nu+1,nw+nu+nsi\\ j=nw+nu+1,nw+nu+nsi}}=$$

$$\frac{b}{12}\int_0^L\beta_{i-nw-nu}\beta_{j-nw-nu}t^3\left(x\right)\rho\left(x\right)dx$$

Rest all other elements of the mass matrix $[M_{ij}]$ are zero.

Both these matrices [K] and [M] are known parameters at this point of dynamic analysis. All the stiffness matrix elements can be completely evaluated on the basis of the converged static solution, while the elements of the mass matrix are derived from the problem definition. It is to be noted that the system governing equations given by Equation (13) represent a standard eigenvalue problem, solution to which is obtained by using Matlab's intrinsic solver. Natural frequencies of the pre-stressed system are provided by the square root of the eigenvalues. Mode shapes of the vibrating system are obtained from the eigenvectors.

3. RESULT AND DISSCUSSION

In the present article, free vibration analysis of transversely loaded non-uniform AFG Timoshenko beam resting on elastic foundation is performed. The static analysis inflicting large deformation under transverse uniform loading is geometrically nonlinear in nature, while an eigenvalue analysis is undertaken to determine the loaded natural frequencies of the system. The beam in the present analysis is considered to be non-uniform with variable thickness. Linear tapering of the thickness is considered in the axial direction according to the expression, $t(x) = t_0 (1-\alpha x/L)$. Here, α is the taper parameter. The geometric dimensions of the beam for length (L) and width (b) are taken as 0.2 m and 0.02 m respectively. Length-to-thickness ratio (L/t_0) and taper parameter (α) are taken as 20 and 0.2, respectively.

Three different material models are selected for the AFG beam, where gradation of material properties (i.e. elastic modulus and density) are considered in the axial direction. The details of the property variations in terms of expressions are provided in Table 2. These expressions are presented with respect to a normalized axial coordinate defined as, $\xi = x/L$. It is important to note that all the numerical evaluations are carried out in the normalized domain. E_0 and ρ_0 are elastic modulus and mass density of the root side of the beam and taken as 210 GPa and 7850 kg/m^3, respectively, which resemble the property of structural steel. From the Table 2, it is clear that material 1 is the case of homogeneous beam which is used in the present study only for comparison purpose to effectively detect the effect of gradation. The gradation of elastic modulus ($E(\xi)$) and density ($\rho(\xi)$) in the axial direction (ξ) for different material models are also shown in Figure 2(a) and Figure 2(b) separately in a normalized plane. It is important to note that Poisson ratio *(μ)* is taken as constant throughout the entire analysis which is 0.3.

Table 2. Three different Material Model:

Material Model	Elastic Modulus	Density
Material 1	$E(\xi)= E_0$	$\rho(\xi)= \rho_0$
Material 2	$E(\xi)= E_0(1+\xi)$	$\rho(\xi)= \rho_0(1+\xi+\xi^2)$
Material 3	$E(\xi)= E_0\,exp(\xi)$	$\rho(\xi)= \rho_0\,exp(\xi)$

Three different flexural boundary conditions, namely, CC, CS and SS, of the beam are selected, while for in-plane boundary conditions it is assumed that all the ends are immovable. At the clamped (C)

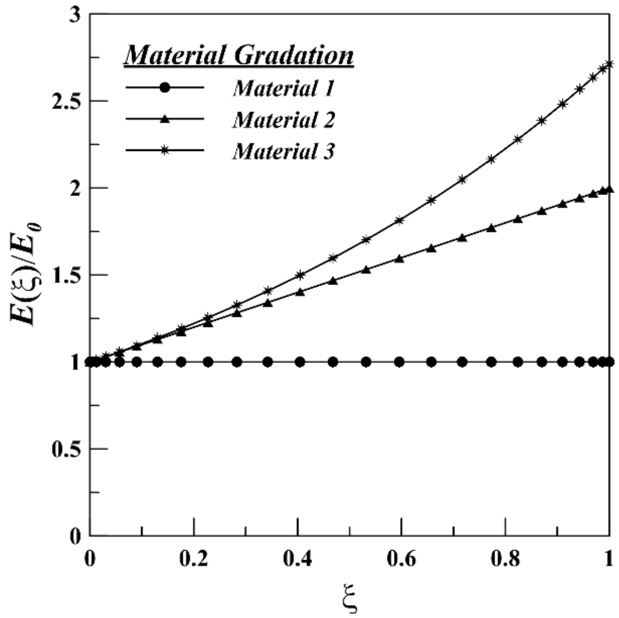


Figure 2(a). Gradation of elastic modulus ($E(\xi)/E_0$) in the axial direction (ξ) for different material model.

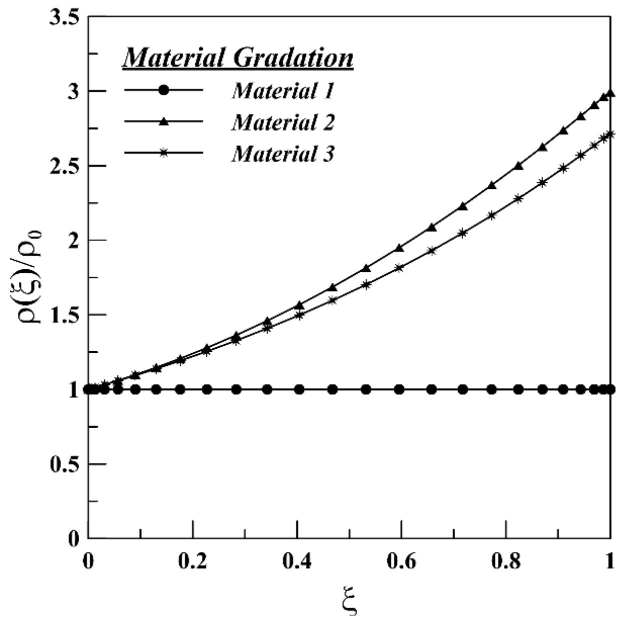
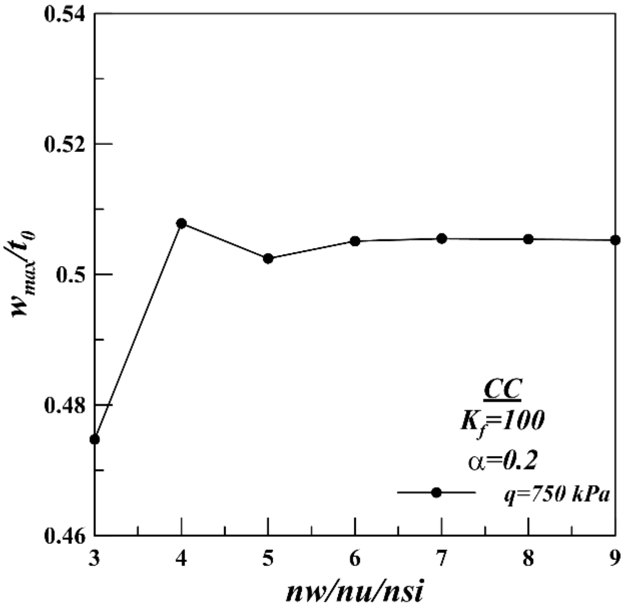


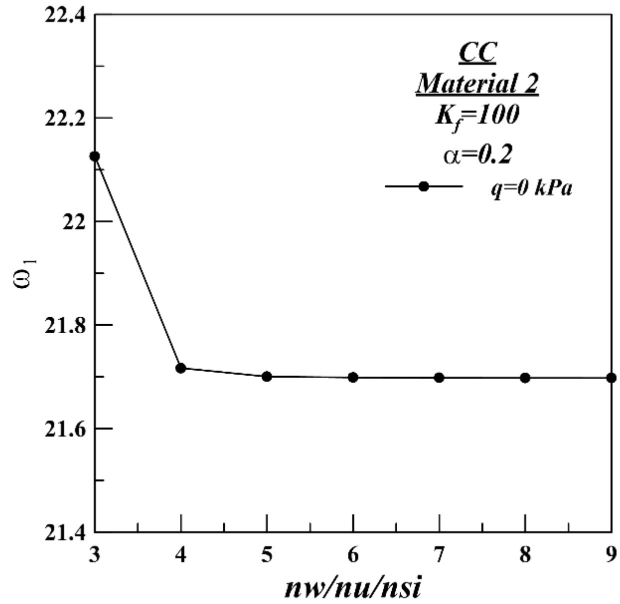
Figure 2(b). Gradation of density ($\rho(\xi)/\rho_0$) in the axial direction (ξ) for different material model.

end the rotational field has a zero value, whereas it has a non-zero value at the simply supported (S) end. The details of the boundary conditions along with the corresponding choice for the start functions are provided in Table 1. Higher order functions to complete the orthogonal set are generated by implementing a numerical scheme for Gram–Schmidt orthogonalization principle, where the start function and the number of functions to be generated serve as inputs.

The appropriate number of functions to be used is decided after a careful convergence study and these results are discussed in the next paragraph. In practical applications beams are



(a)



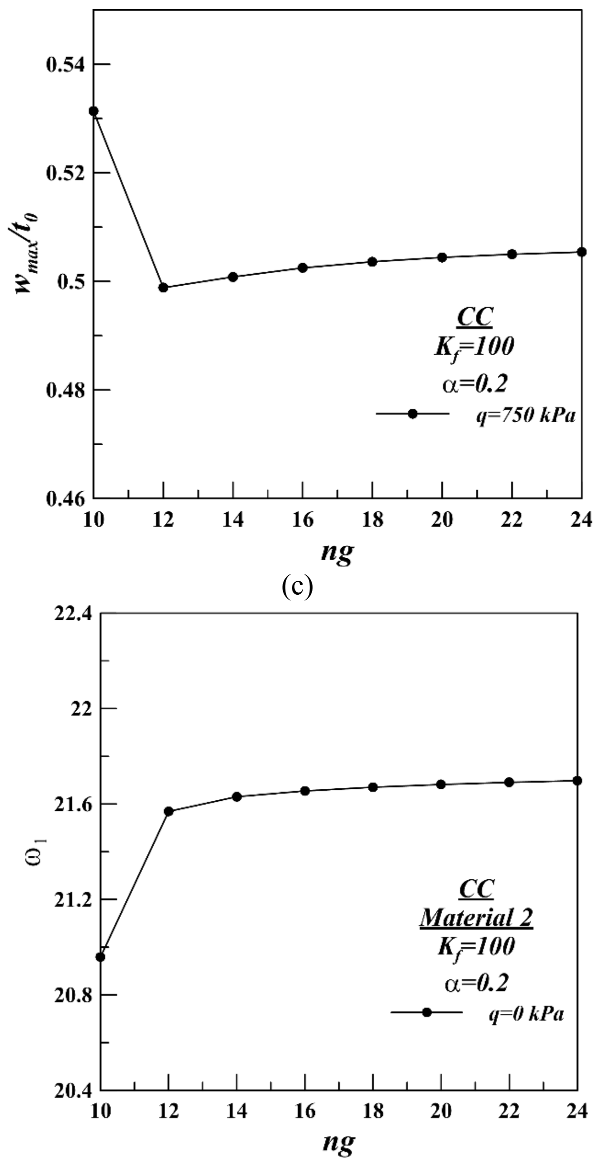
(b)

Figure 3. (a)-(b) Convergence studies for no of orthogonal functions (*nw=nu=nsi*) corresponding to static and free vibration analysis

seldom supported by classical boundary conditions. Either they are supported by elastic restrain at the ends or rested on elastic foundation. To make the problem more realistic, the beam under consideration is taken to be resting on elastic foundation. Four different value of the non-dimensional foundation stiffness [$K_f = k_f\left[\left(L/t_0\right)^3/E_0 b\right]$] is consider here.

These values are 0, 10, 10^2 and 10^3 respectively. Here, k_f is the dimensional value of stiffness.

In order to create the reference points for numerical evaluation within the computational domain, number of gauss points are generated along the length of the beam. Choice of number of gauss points (*ng*) is an important issue, as it influences the outcome of the numerical scheme. Hence, detailed convergence study is necessary for selection of proper number of gauss points. This convergence study, along with the one carried out for determining the number of functions, is carried out on an AFG (Material 2) beam with clamped ends and supported



(c)

(d)

Figure 3. (c)-(d) Convergence studies for gauss points (*ng*) corresponding to static and free vibration analysis.

on an elastic foundation having K_f = 100. There are two aspects to these studies, first, comparison of normalised maximum static deflection and second, comparison of dimensional fundamental frequency with respect to the relevant parameters. For the static scenario, intensity of the distributed load is taken as 750 kPa, while, for the dynamic case, no load condition is assumed. The results of the study are presented in Figure 3 and from these figures, number of gauss points *(ng)* and number of orthogonal functions *(nw=nu=nsi)* are selected as 24 and 8, respectively.

Table 3. Comparisons of first four dimensionless $\left(\omega = \Omega L^2 \sqrt{\rho_0 A_0 / E_0 I_0}\right)$ natural frequencies of graded and non-uniform Timoshenko CC beam for different *n* value.

Literatures	ω	n=1	n=2	n=3	n=4
Shahba et al [12]	1	-	12.47	-	-
Huang et al [13]		12.68	12.46	12.38	12.36
Present study		12.65	12.43	12.34	12.33
Shahba et al [12]	2	-	26.42	-	-
Huang et al [13]		26.49	26.38	26.32	26.31
Present study		26.43	26.32	26.26	26.25
Shahba et al [12]	3	-	43.09	-	-
Huang et al [13]		42.64	42.96	43.08	43.1
Present study		42.54	42.86	42.98	43.03
Shahba et al [12]	4	-	59.68	-	-
Huang et al [13]		58.67	59.40	59.70	59.82
Present study		58.51	59.23	59.52	59.64

The present methodology and analysis is validated with the results of previously published articles of Shahba et al [12] and Huang et al [13] for fully axial graded (ZrO_2 - Al) and non-uniform Timoshenko beam without elastic foundation. The comparison of linear dimensionless natural frequencies $\left(\omega = \Omega L^2 \sqrt{\rho_0 A_0 / E_0 I_0}\right)$ for first four mode are tabulated in Table 3 for CC end conditions. For this purpose, the length (L) and moment of inertia to cross-section ratio (I_0/A_0) at root side of the beam are used as 1.0 m and 0.01, respectively. The material gradation along the axial direction follows the expressions, $E(\xi)= E_0+(E_1-E_0)(\xi^n)$ and $\rho(\xi)= \rho_0+(\rho_1-\rho_0)(\xi^n)$. The material properties used for comparison are as follows, E_0 = 200 GPa, ρ_0 = 5700 kg/m^3, E_1 = 70 GPa and ρ_1 = 2702 kg/m^3. The non-uniform pattern of the beam is considered according to the expression, $t(\xi)= t_0(1-\alpha\xi)$ with taper parameter, α = 0.1. From the above comparison, it can be observed that the current results match satisfactorily with the established results.

The effect of the taper parameter and the foundation stiffness values on natural frequencies are shown in Table 4. The linear dimensionless natural frequencies $\left(\omega_1 = \Omega L^2 \sqrt{\rho_0 A_0 / E_0 I_0}\right)$ for CC, CS and SS end conditions and different foundation stiffness values are tabulated in Table 4(a), Table 4(b) and Table 4(c), respectively.

Table 4. Dimensionless $\left(\omega_1 = \Omega L^2 \sqrt{\rho_0 A_0 / E_0 I_0}\right)$ fundamental frequency of graded Timoshenko beam on elastic foundation for different stiffness value.

Table 4(a): CC End condition

Taper Parameter (α)	Foundation Stiffness (K_f)	Material 1	Material 2	Material 3
		ω_1	ω_1	ω_1
0	0	21.92	20.06	22.06
	10	22.46	20.40	22.39
	10^2	26.83	23.22	25.16
	10^3	53.62	41.85	44.04
0.2	0	19.74	17.91	19.70
	10	20.40	18.32	20.10
	10^2	25.63	21.70	23.43
	10^3	55.30	42.47	44.55

Table 4(b): CS End condition

Taper Parameter (α)	Foundation Stiffness (K_f)	Material 1	Material 2	Material 3
		ω_1	ω_1	ω_1
0	0	15.24	13.10	14.21
	10	16.01	13.57	14.68
	10^2	21.72	17.25	18.38
	10^3	51.25	37.50	39.27
0.2	0	14.12	12.10	13.16
	10	15.05	12.67	13.73
	10^2	21.69	16.96	18.04
	10^3	53.90	39.17	40.97

Table 4(c): SS End condition

Taper Parameter (α)	Foundation Stiffness (K_f)	Material 1	Material 2	Material 3
		ω_1	ω_1	ω_1
0	0	9.83	9.01	9.80
	10	10.98	9.73	10.51
	10^2	18.33	14.68	15.47
	10^3	49.92	37.57	38.96
0.2	0	8.86	8.11	8.81
	10	10.25	8.99	9.67
	10^2	18.57	14.70	15.41
	10^3	52.34	39.42	40.84

It is observed from the above table that for all the cases with the increase in stiffness of the foundation the natural frequency increases, which is quite an expected outcome. This increment of frequency is due to reason that stiffer foundation makes the system more rigid. It is also observed that the rigidity of the beam edges and consequently the boundary conditions significantly affect the free vibration response. That's why, the natural frequency of CC beam is the highest and for SS beam, it is the lowest for a fixed value of foundation stiffness. It is important to note that taper pattern value '0' indicates the case of uniform beam. It is observed that when taper pattern has a definite value (i.e. case of non-uniform beam) the natural frequencies have higher values as compared to the natural frequencies of uniform beam. This is due to the reason of gradual decrement of system mass as natural frequency is inversely proportional to it.

Variation of natural frequencies with respect to changes in transverse pre-load value is the main focus of the study. However, as the transverse load produces a given static deflection, the results are presented in the normalised frequency vs. normalised maximum deflection plane. The graphical representation, where the ordinate represents dimensionless amplitude (w_{max}/t_0) and abscissa represents the dimensionless frequency (ω_{nf}/ω_1), is akin to a backbone curve. The formulation has been developed in such a way that the system can be equivalently assumed to represent a large amplitude vibration scenario. Such an approach is justified on the basis of the assumption that large amplitude vibration analysis of a nonlinear system may be considered equivalent to a case where the system undergoes small oscillations about the deflected configuration of same large amplitude [32]. It is important to note that the fundamental frequencies (ω_1) at no load condition, which are already obtained and tabulated in Table 4, are used here to normalize nonlinear frequencies (ω_{nf}/ω_1).

The effect of the foundation stiffness on backbone curves are shown in Figure 4 for three different boundary conditions i.e. CC, CS and SS, respectively. Plots are generated considering four dimensionless foundation stiffness values, i.e. 0, 10, 10^2 and 10^3 respectively. It is to be noted that foundation stiffness value '0' represents the case where the beam is without any foundation. Two different material models are also considered to detect the effect more accurately. Backbone 1 and backbone 2 are plotted in each case considering taper parameter (α) as 0.2. From the figure, it is cleared that, for CC beam, with the increase of the foundation stiffness the curves become straighter for both material 1 and material 2. Similar trends can be seen for CS and SS beam as well. It is also observed in all the situations that increase in maximum normalised deflection corresponds to increase in the normalised loaded natural frequency as well, which indicates a hardening type behaviour.

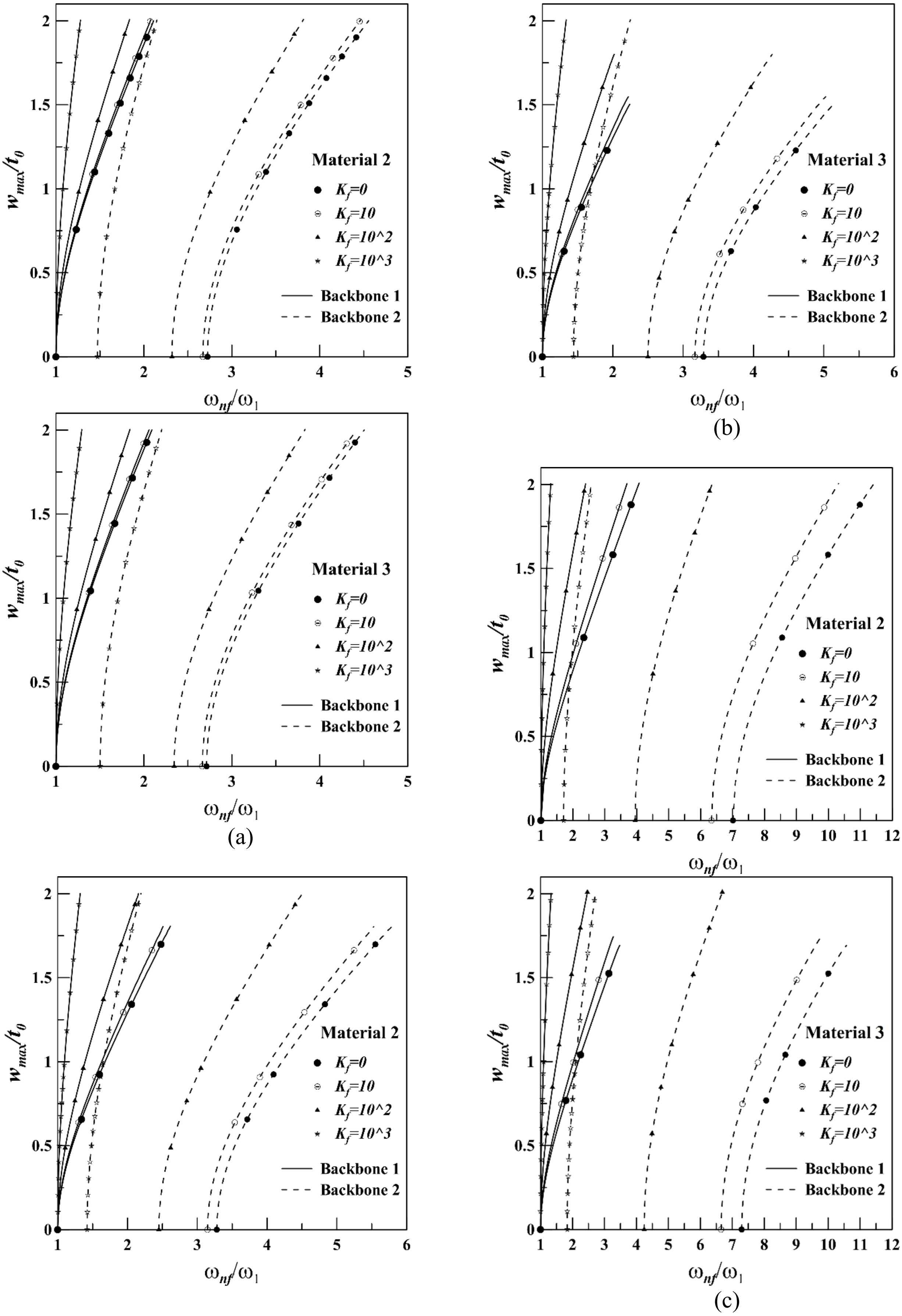


Figure 4. Effect of foundation stiffness on backbone curve of AFG non-uniform Timoshenko beam for (a) CC, (b) CS and (c) SS end conditions respectively.

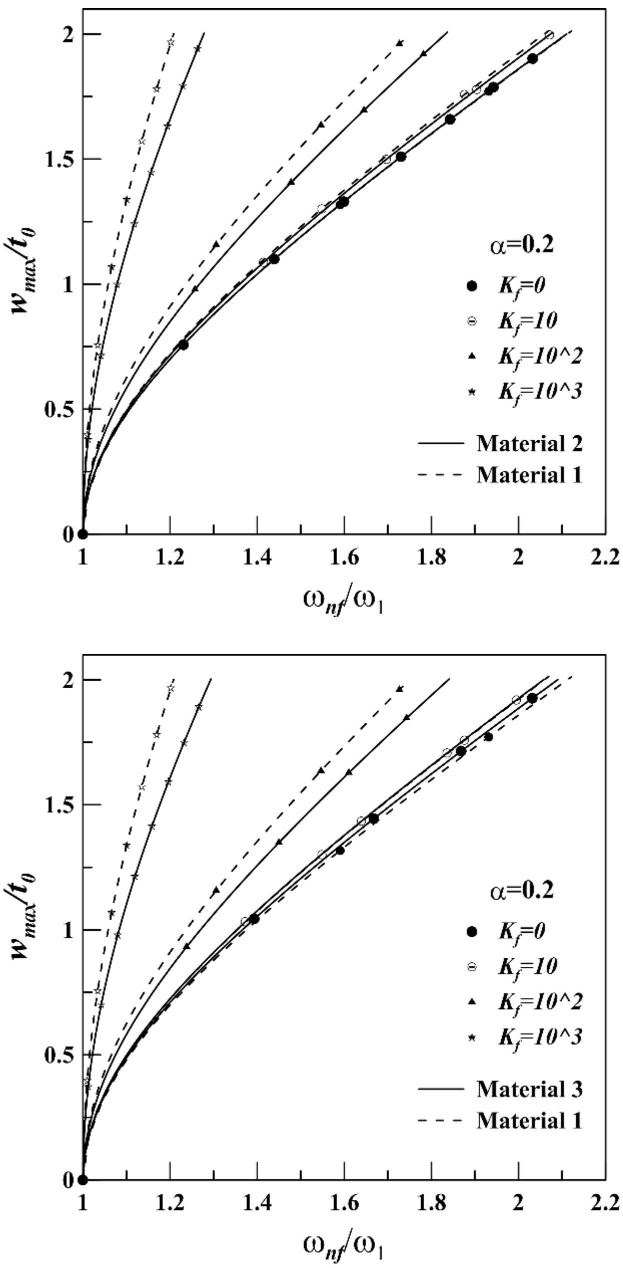


Figure 5. Effect of the material gradation on backbone curve of AFG non-uniform Timoshenko beam for CC end condition.

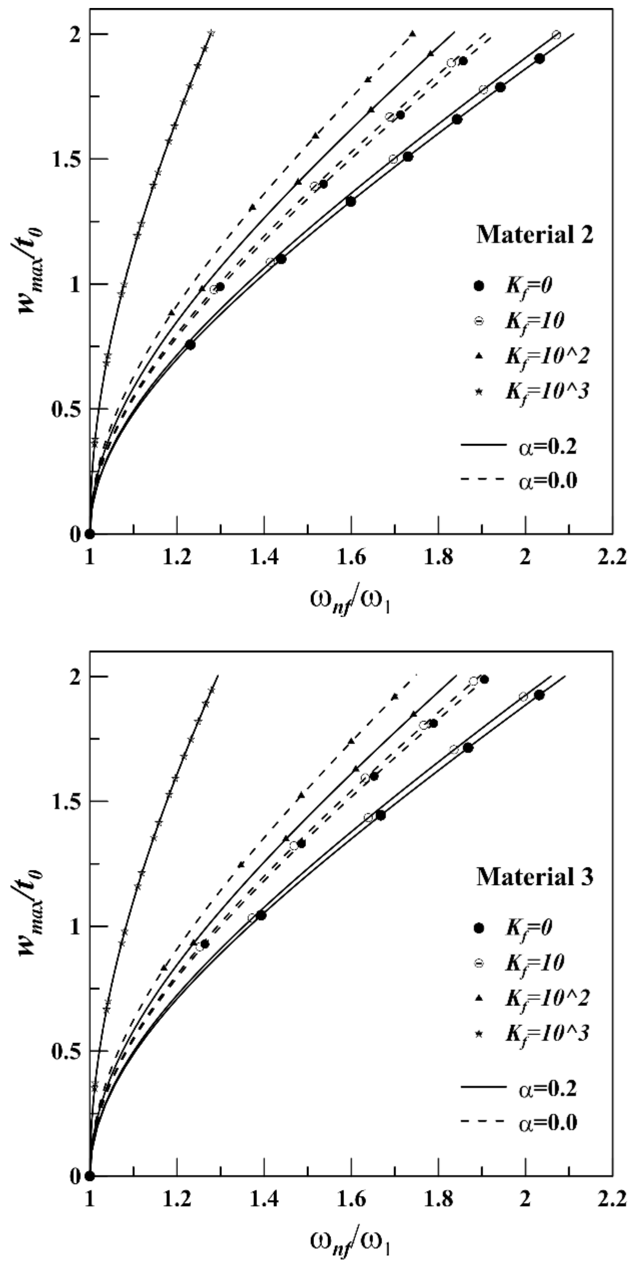


Figure 6. Effect of the taper parameter on backbone curve of AFG non-uniform Timoshenko beam for CC end condition.

The effects of the material gradation are shown in Figure 5 through the representation of the plot for loaded fundamental frequency. The effects of material 2 over material 1 and material 3 over material 1 are plotted in two different sub plots. Here, material 1 is clearly the case of homogeneous beam without any material gradation. For this purpose, a clamped non-uniform Timoshenko beam is considered with taper parameter (α) 0.2. It is observed that for small foundation stiffness value the effect of the material gradation is not that prominent but at higher foundation stiffness values there is substantial difference between the two curves. Similar observations can be made for material 3 and material 1 as well. This is due to the reason that in the present study the material gradation is considered in such a way that the material property values continuously increase along the length, while the cross-section reduces.

The effect of the taper parameter is shown in Figure 6 through the representation of the curve corresponding to the loaded natural frequency for the 1st mode. For this purpose, again a CC Timoshenko beam is chosen. It is observed that for that for material 2 at higher foundation stiffness value the effect of the taper parameter is not that noticeable but at lower foundation stiffness value there is difference between the two curves. Similar effect can be seen for material 3 as well. This decrement of frequency is due to the

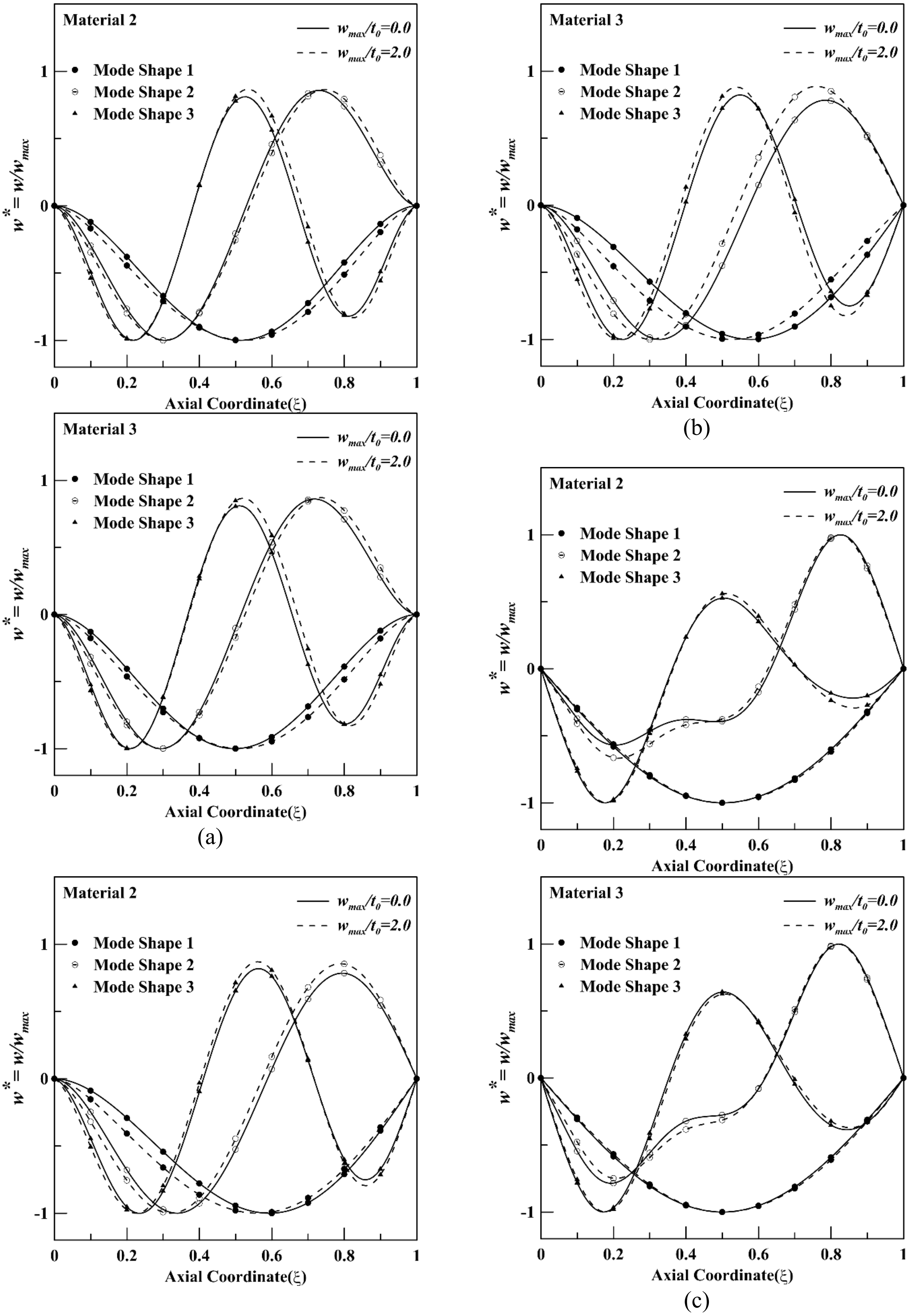


Figure 7. First three mode shape of AFG non-uniform Timoshenko beam for (a) CC, (b) CS and (c) SS end condition respectively

softening effect introduced by the decrease in cross-sectional area and moment of inertia.

The free vibration analysis as represented by Equation (13) is a standard eigenvalue problem and its solution provides not only information regarding the natural frequencies but also eigenvectors corresponding to the eigenvalues. In fact, the unknown parameters associated with Equation (13) denote the eigenvectors in matrix form and the contribution of individual spatial functions on the vibration modes. However, it is important to note that the stiffness matrix in this eigenvalue analysis corresponds to the converged large deflection static solution. So, equivalence may be drawn between large amplitude free vibration of a nonlinear system and its free vibration analysis, subjected to a static load producing same magnitude of large amplitude deflection [32, 28]. So, the evaluated mode shapes can be considered as corresponding to large amplitude vibration about the undeformed equilibrium position.

Linear (w_{max}/t_0 =0.0) and nonlinear (w_{max}/t_0 =2.0) mode shapes for the first three modes corresponding to CC, CS and SS boundary conditions are shown in Figures 7. These results are furnished for an AFG (both material 2 and 3) tapered (α = 0.2) Timoshenko beam on elastic foundation (K_f=100). Changes in the mode shape in between the linear and nonlinear one are apparent from the figures.

It should be mentioned that the analysis performed in the present paper is general in nature. But the mathematical formulation and solution methodology are robust enough to extend it to other application-oriented areas, such as damage identification in Timoshenko beams, application as aircraft wings [33, 34] etc.

4. CONCLUSIONS

In the present paper, a non-uniform AFG Timoshenko beam on elastic foundation under pre-loaded condition is analysed to determine the natural frequencies. Three different boundary conditions, which are combinations of clamped and simply supported edges, are selected. The elastic foundation, in the present study, is idealized as a set of parallel linear spring of constant stiffness and various values of the foundation stiffness are considered. The primary objective is to find out the effect of loading on the vibration frequencies of the system. The mathematical formulation is such that it sub-divides the problem into two distinct parts. At the initial stage the geometrically nonlinear static problem is solved through an iterative scheme with relaxation. Subsequently, the free vibration problem is formulated as an eigenvalue analysis with statically converged stiffness matrix as an input. These problems are formulated using appropriate energy principles. The static analysis is based on total minimum potential energy principle, while the dynamic analysis is based on Hamilton's principle. The methodology of the mathematical formulation is general in nature. It has enough flexibility of solving with other different type of loading pattern, elastic foundation pattern, gradation pattern, taper pattern and end conditions as well. Results generated from the proposed method are compared with previously published results and a certain degree of accuracy is observed in between the two sets of results. Overall, the present methodology and solution procedure are successfully validated, albeit for a system with reduced complexity (as the elastic foundation is not present in the validation problem). New results are furnished for an AFG Timoshenko beam in the normalised loaded natural frequency vs. normalised maximum deflection of the system. These results are capable of serving as the benchmark results for future reference. These pre-stressed natural frequency values exhibit hardening type nonlinearity.

REFERENCES

[1] El-Ashmawy A.M., Kamel M.A., Elshafei M.A., Thermo-mechanical analysis of axially and transversally function graded beam, *Composites Part B,* Vol. 102, 2016, pp. 134-149.

[2] Zhao Y., Huang Y., Guo M., A novel approach for free vibration of axially functionally graded beams with non-uniform cross-section based on Chebyshev polynomials theory, *Composite Structures,* Vol. 168, 2017, pp. 277–284.

[3] Sari M., Shaat M., Abdelkefi A., Frequency and mode veering phenomena of axially functionally graded non-uniform beams with nonlocal residuals, *Composite Structures,* Vol. 163, 2017, pp. 280–292.

[4] Kiani K., Thermo-elasto-dynamic analysis of axially functionally graded non-uniform nanobeams with surface energy, *International Journal of Engineering Science,* Vol. 106, 2016, pp. 57–76.

[5] Huang Y., Zhang M., Rong H., Buckling analysis of axially functionally graded and non-uniform beams based on Timoshenko theory, *Acta Mechanica Solida Sinica,* Vol. 29(2), 2016, pp. 200-207.

[6] Chen D.Q., Sun D.L., Li X.F., Surface effects on resonance frequencies of axially functionally graded Timoshenko nanocantilevers with attached nanoparticle, *Composite Structures,* Vol. 173, 2017, pp. 116–126.

[7] Sarkar K., Ganguli R., Closed-form solutions for axially functionally graded Timoshenko beams having uniform cross-section and fixed–fixed boundary condition, *Composites: Part B,* Vol 58, 2014, pp. 361–370.

[8] Shafiei N., Kazemi M., Ghadiri M., Comparison of modelling of the rotating tapered axially functionally graded Timoshenko and Euler–Bernoulli microbeams, *Physica E,* Vol. 83, 2016, pp. 74–87.

[9] Huang Y., Wang T., Zhao Y., Wang P., Effect of axially functionally graded material on whirling frequencies and critical speeds of a spinning Timoshenko beam, *Composite Structures,* Vol. 192, 2018, pp. 355–367.

[10] Wang Y., Wu D., Thermal effect on the dynamic response of axially functionally graded beam subjected to a moving

harmonic load, *Acta Astronautica,* Vol. 127, 2016, pp. 171–181.
[11] Calim F.F., Transient analysis of axially functionally graded Timoshenko beams with variable cross-section, *Composites Part B,* Vol. 98, 2016, pp. 472-483.
[12] Shahba A., Attarnejad R., Marvi M.T., Hajilar, S., Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions, *Composites: Part B*, Vol. 42, 2011, pp. 801–808.
[13] Huang Y., Yang L.E., Luo Q.Z., Free vibration of axially functionally graded Timoshenko beams with non-uniform cross-section, *Composites: Part B,* Vol. 45, 2013, pp. 1493–1498.
[14] Rajasekaran S., Free vibration of centrifugally stiffened axially functionally graded tapered Timoshenko beams using differential transformation and quadrature methods, *Applied Mathematical Modelling,* Vol. 37, 2013, pp. 4440–4463.
[15] Yas M.H., Samadi N., Free vibrations and buckling analysis of carbon nanotube-reinforced composite Timoshenko beams on elastic foundation, *International Journal of Pressure Vessels and Piping,* Vol 98, 2012, pp. 119-128.
[16] Komijani M., Esfahani S.E., Reddy J.N., Liu Y.P., Eslami, M.R., Nonlinear thermal stability and vibration of pre/post-buckled temperature and microstructure-dependent functionally graded beams resting on elastic foundation, *Composite Structures,* Vol. 112, 2014, pp. 292–307.
[17] Mohanty S.C., Dash R.R., Rout T., Parametric instability of a functionally graded Timoshenko beam on Winkler's elastic foundation, *Nuclear Engineering and Design,* Vol. 241, 2011, pp. 2698– 2715.
[18] Tossapanon P., Wattanasakulpong N., Stability and free vibration of functionally graded sandwich beams resting on two-parameter elastic foundation, *Composite Structures,* Vol 142, 2016, pp. 215–225.
[19] Deng H., Chen K.D., Cheng W., Zhao S.G., Vibration and buckling analysis of double-functionally graded Timoshenko beam system on Winkler-Pasternak elastic foundation, *Composite Structures,* Vol. 160, 2017, pp. 152–168.
[20] Arefi M., Zenkour A.M., Wave propagation analysis of a functionally graded magneto-electro-elastic nanobeam rest on visco-Pasternak foundation, *Mechanics Research Communications,* 2017, Vol 79, pp. 51–62.
[21] Arefi M., Zenkour A.M., Analysis of wave propagation in a functionally graded nanobeam resting on visco-Pasternak's foundation, *Theoretical & Applied Mechanics Letters,* Vol. 7, 2017, pp. 145–151.
[22] Yan T., Kitipornchai S., Yang J., He X.Q., Dynamic behaviour of edge-cracked shear deformable functionally graded beams on an elastic foundation under a moving load, *Composite Structures,* Vol. 93, 2011, pp. 2992–3001.
[23] Esfahani S.E., Kiani Y., Eslami M.R., Non-linear thermal stability analysis of temperature dependent FGM beams supported on non-linear hardening elastic foundations, *International Journal of Mechanical Sciences,* Vol. 69, 2013, pp. 10–20.
[24] Calim F.F., Free and forced vibration analysis of axially functionally graded Timoshenko beams on two-parameter viscoelastic foundation, *Composites Part B,* Vol. 103, 2016, pp. 98-112.
[25] Huang Y., Luo Q.Z., A simple method to determine the critical buckling loads for axially inhomogeneous beams with elastic restraint, *Computers and Mathematics with Applications,* Vol. 61, 2011, pp. 2510–2517.
[26] Lohar H., Mitra A., Sahoo S., Geometric nonlinear free vibration of axially functionally graded non-uniform beams supported on elastic foundation, *Curved and Layered Structures*, Vol. 3 (1), 2016, pp. 223–239.
[27] Lohar H., Mitra A., Sahoo S., Natural frequency and mode shapes of exponential tapered AFG beams on elastic foundation, *International Frontier Science Letters,* Vol. 9, 2016, pp. 9-25.
[28] Mitra A., Sahoo P., Saha K.N., Free vibration analysis of initially deflected stiffened plates for various boundary conditions, *Journal of Vibration and Control,* Vol. 17(14), 2011, pp. 2131–2157.
[29] Paul A., Das D., Free vibration analysis of pre-stressed FGM Timoshenko beams under large transverse deflection by a variational method, *Engineering Science and Technology, an International Journal,* Vol. 19, 2016, pp. 1003–1017.
[30] Shames I.H., Dym C.L., *Energy and finite element methods in structural mechanics*, New Age International Publishers, Delhi, 2009.
[31] Kumar S., Mitra A., Roy H., Geometrically nonlinear free vibration analysis of axially functionally graded taper beams, *Engineering Science and Technology, an International Journal,* Vol. 18, 2015, pp. 579-593.
[32] Crandall S.H., *Engineering analysis*, McGraw-Hill, New York, 1956.
[33] Gillich G-R., Praisach Z-I., Onchis-Moaca, D., Finite Element Method based Algorithm for Localization of Damages in Cantilever Beams, *Romanian Journal of Acoustics and Vibration*, Vol 7(2), 2010, pp. 26-32.
[34] Bennamia I., Badereddine A-E., Yahia Cherif M., Zebbiche T., Frequency Response Analysis of Composite Aircraft Wing Using a Finite Element Euler-Bernoulli Beam Model, *Romanian Journal of Acoustics and Vibration*, Vol 15(1), 2018, pp. 115-121.