Modal And Frequency Response Analysis of Composite Unidirectional Thick Beam Using Finite Element Timoshenko and Euler-Bernoulli Beam Models

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Abstract: - The Timoshenko beam and the Euler-Bernoulli beam models are adopted in this purpose study for modal and frequency response analysis of unidirectional composite thick cantilever beam (Single thick composite layer forms the beam). The shear deformation effect is the main parameter taken in this study where Timoshenko beam model results are compared to those of the Euler-Bernoulli beam model. The first part of this study is consecrated to determinate the mode shapes and their associate natural frequencies of the cantilever beam for various shear deformation values by using the finite element method. The mode shapes and their natural frequencies results for the unidirectional layer composite cantilever beam are validated with DSM (Dynamic Stiffness Method) published results. The frequency response analysis of an impulse force of both models is illustrated, in the second part using a MATLAB@ program.

Keywords: -Timoshenko beam, Euler-Bernoulli beam, Composite Beam, Modal Analysis, Frequency Response

1. INTRODUCTION

The Timoshenko theory and Euler-Bernoulli theory are the most theories used to study the dynamic behavior or vibration motion of beams.

The Euler-Bernoulli model is used in free vibration analysis of metallic beams since the 18th century, where the Timoshenko beam model is developed in the 20th century [1].

In 1972, A theoretical analysis of the vibration of unidirectional fiber reinforced, composite beams paper is given by Teoh and Hung [2].

Several researches use the Dynamic Stiffness Method (DSM) to study the free vibration analysis of composite beams: J. R Banerjee gives an exact expression for the frequency equation and mode shape formulae of composite Timoshenko beams. An exact dynamic stiffness matrix is presented for a composite beam includes the effects of shear deformation and rotatory inertia with applications to composite Timoshenko beam developed by Banerjee and Williams [3]. Su and Banerjee give recent development of dynamic stiffness method for free vibration of functionally graded Timoshenko beams [4]. A Dynamic stiffness formulation and free vibration analysis of centrifugally stiffened Timoshenko beams[5] and free vibration of axially loaded composite Timoshenko beams using the dynamic stiffness matrix method[6] are developed by J. R Banerjee.

The effect of elastic foundation on free vibration of initially deflected non-uniform axially functionally graded (AFG) thick beam on elastic foundation on the basis of Timoshenko beam theory is studied by Lohar et al.[7]. The shape function method is adopted by Zhao [8] to obtain the frequency and mode shape equations of the free vibrations of Euler-Bernoulli beams with an arbitrary number of intermediate elastic supports, concentrated masses, and non-conventional boundary conditions under an axial force.

The finite element method is used by Deghboudj and Boukhedena to study the free vibration analysis and control passive of Aluminum alloy plate with damping orthotropic patches [9] and used by Gillich et al to localize damages in cantilever beams [10].

Kwon and Bang develop a computer programs in MATLAB to study the Finite Element Method and the Fast Fourier Transform (FFT) results [11].

In the purpose study, The Timoshenko beam and Euler Bernoulli beam (no effects of transverse shear deformation) theories are adopted for modal analysis and frequency response analysis of a single layer composite beam. A MATLAB computer programs are developed to show the influence of the transverse shear deformation effects on modal analysis and frequency response analysis of clamped-free composite beam using the Fast Fourier Transform (FFT) results at the tip of the beam.

A single glass/epoxy thick layer model the section of the beam (Figure 1).

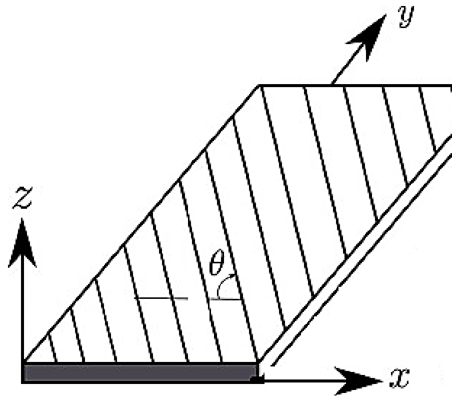


Figure 1. A single-layer composite beam model

2. THE FINITE ELEMENT MODELS

2.1. The Euler-Bernoulli beam Model

The governing partial differential equations of motion for the coupled bending-torsional free vibration of composite beam are [11]:

$$EI\frac{\partial^4 w}{\partial x^4}+K\frac{\partial^3 \Psi}{\partial x^3}+m\frac{\partial^2 w}{\partial t^2}=0 \qquad (1.a)$$

$$GJ\frac{\partial^2 \Psi}{\partial x^2}+K\frac{\partial^3 w}{\partial x^3}-I_\alpha\frac{\partial^2 \Psi}{\partial t^2}=0 \qquad (1.b)$$

where:

- EI is the bending stiffness,
- GJ is the torsional stiffness
- K the stiffness of bending-torsion coupling
- m, I_α is the mass per unit in length and the moment of inertia about the axis elastic.

The finite element Euler-Bernoulli beam model is developed by Bennamia for modal and frequency response analysis where the finite element method is developed to deduce the mass and stiffness matrices of the laminated composite beam [12].

2.2. The Timoshenko beam Model

The Timoshenko beam model developed in this section is based on the assumption that the plan normal to the beam axis before deformation does not remain to the beam axis after deformation [11] as seen in figure 2, so the transverse shear deformation is included.

The governing partial differentialequations of motion for the coupled bending-torsional free natural vibration of the composite Timoshenko beam are given by [1]:

$$EI\frac{\partial^2 \theta}{\partial y^2}+kAG\left(\frac{\partial v}{\partial y}-\theta\right)+K\frac{\partial^2 \Psi}{\partial y^2}-\rho I\frac{\partial^2 \theta}{\partial t^2}=0 \qquad (2.a)$$

$$kAG\left(\frac{\partial^2 v}{\partial y^2}-\frac{\partial \theta}{\partial y}\right)-\rho\frac{\partial^2 v}{\partial t^2}=0 \qquad (2.b)$$

$$GJ\frac{\partial^2 \Psi}{\partial y^2}+K\frac{\partial^2 \theta}{\partial y^2}-I_\alpha\frac{\partial^2 \Psi}{\partial t^2}=0 \qquad (2.c)$$

where v, Ψ and θ represent the movement in bending, torsion and angle of rotation, respectively. EI is the bending rigidity, GJ is the torsional rigidity, K is the bending-torsion coupling rigidity and kAG is the shear rigidity of the beam. ρ is the mass density, I_α is the moment of inertia about the axis elastic and I is the second moment of area of the beam cross section.

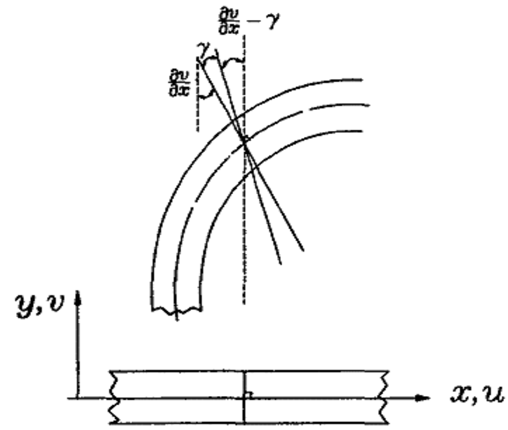


Figure 2. Timoshenko beam model [11]

2.2.1 The Kinetic Energy of the Timoshenko beam model

The kinetic energy of bending-torsional layered composite beam given by [12]:

$$T=\frac{1}{2}\int \rho(\{\dot{w}\})^2 dxdydz \qquad (3)$$

where w and Ψ represent the movement in bending and torsion, respectively, ρ is the mass density and $x_\alpha$is the geometric coupling (distance between the mass axis and elastic axis of the wing).

The kinetic energy can be written as [14][18]:

$$T = \frac{1}{2}\{\ddot{q}\}^t[M_e]\{\ddot{q}\} \quad (4)$$

where
$[M_e]$: Beam element mass matrix

2.2.2 The Potential Energy of the Timoshenko beam model

The potential energy of bending-torsional layered composite beam given by [11]:

$$U = U_{coupled} + U_{decoupled}$$

$$U_{decoupled} = U_{bending} + U_{shear}$$

The shear and bending energies are given by [11]:

$$U_{bending} = \frac{b}{2}\int_0^l \left(\frac{\partial\theta}{\partial x}\right)^T EI \left(\frac{\partial\theta}{\partial x}\right) dx, \quad (5.a)$$

$$U_{shear} = \frac{\mu}{2}\int_0^l \left(-\theta + \frac{\partial v}{\partial x}\right)^T GA \left(-\theta + \frac{\partial v}{\partial x}\right) dx, \quad (5.b)$$

$$U_{coupled} = \frac{1}{2}\int K \left(\frac{d^2 w}{dx^2}\right)\left(\frac{d\Psi}{dx}\right) dx, \quad (5.c)$$

where
μ is the correction factor for shear energy.

The kinetic energy can be written as [14][18]:

$$U = \frac{1}{2}\{q\}^t[K_e]\{q\} \quad (6)$$

where
$[K_e]$: Beam element stiffness matrix

2.3. Modal and frequency response Analysis

The governing differential equation of motion of forced system given as [11]:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F\}U = \frac{1}{2}\{q\}^t[K_e]\{q\} \quad (7)$$

The natural frequencies and mode shapes of the beam are calculated by the matrix system obtained by the discrete equations of motion for the global structure:

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \quad (8)$$

$[M]$ is the global mass matrix, $[K]$ is the global stiffness matrix and $\{q\}$ is The displacement vector.

$\{F\}$: vector force

3. THE LAMINATED COMPOSITE THICK BEAM

The composite beam is simulated to clamped-free single unidirectional fiber reinforced composite layer $[\theta = 15°]$, the length of the beam is $L = 0.1905\ m$ the width and thickness of the beam are: $d = 0.0127\ m$ and $t = 0.00318\ m$ [1-2]. The mass per unit of length is $m = 0.0544\ Kg/m$, the mass moment of inertia per unit length is $I_\alpha = 0.7770\ \times 10^6\ Kg\ m$ and the shear rigidity is $kAG = 6343.3\ N$ [1-2].

The bending, torsional and coupled stiffness rigidities are calculated for laminate layer by [13]:

$$EI = d\left(D_{22} - \frac{D_{12}^2}{D_{11}}\right), \quad (9.a)$$

$$GJ = 4d\left(D_{66} - \frac{D_{16}^2}{D_{11}}\right) \quad (9.b)$$

$$K = 2d\left(D_{22} - \frac{D_{12}\, D_{16}}{D_{11}}\right) \quad (9.c)$$

$[D]_{6\times6}$ Called bending stiffness matrix of laminated composite beams [12].

4. RESULTS AND DISCUSSION

4.1 The effective rigidities evaluation for various ply orientation angle

Figures 3 shows the effectives rigidities as a function of ply orientation angles. It appears clearly that the coupling rigidity K equal to zero (decoupled case) for $0°, 90°$ and $\approx 55°$ ply orientation angles.

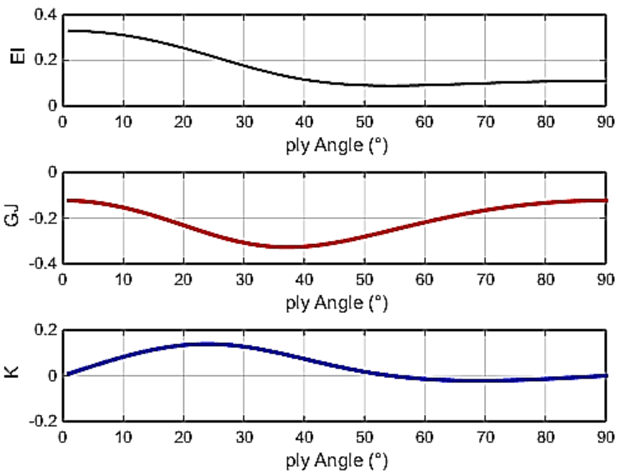


Figure 3. Effective rigidities of single layer cross-section

4.2 Free vibration analysis

4.2.1 Validation

In order to validate the finite element beam models, the four first natural frequencies results obtained from the Euler-Bernoulli and Timoshenko beam models using MATLAB program are compared with those obtained by Banarjee and Williams [3] and Borneman [13] respectively, for the fiber reinforced ply orientation angle $\theta = 15°$.

The effective rigidities for this fiber angle are[1] [3]:

$EI = 0.2865\ N.m^2$, $GJ = 0.1891\ N.m^2$ and $K = 0.1143\ N.m^2$.

4.2.1.1 Euler-Bernoulli beam model

The shear deformation effects are neglected in this model. The table 1 shows the four first frequencies without the effects of shear deformation (Euler-Bernoulli.

Table 1. First four frequencies of Euler-Bernoulli beam model

N°	Euler-Bernoulli Model[Hz]	Reference [3]	Euler-Bernoulli Model [rad/s]	Reference [1]
1	30,81	30,82	193,61	193,62
2	192,72	192,7	1210,93	1210,9
3	537,48	537,4	3377,12	3376,5
4	648,73	648,7	4076,13	4076,1

The first four frequencies obtained by modal analysis using the finite element Euler-beam model are very similar with those obtained by Dynamic Stiffness Method developed by J. R Banerjee [1] and Banerjee and Williams [3].

4.2.1.2 The Timoshenko beam model

1. **Natural frequencies**

The shear deformation effects are given by [1][2][3]:

$$s^2 = \frac{EI}{kAGL^2} \tag{10}$$

The table 2 shows the four first frequencies with the effects of shear deformation ($S = 0.03528$).

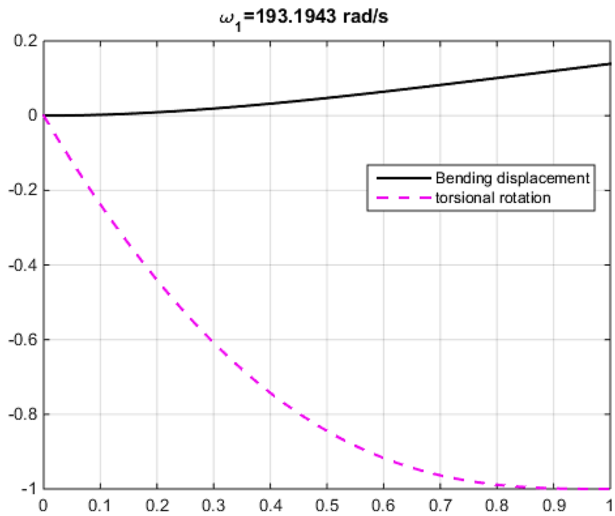
The frequencies results obtained by finite element Timoshenko beam model are similar to Borneman results [13] obtained by DSM approach.

Table 2. First four frequencies of the Timoshenko beam model ($S = 0.03528$)

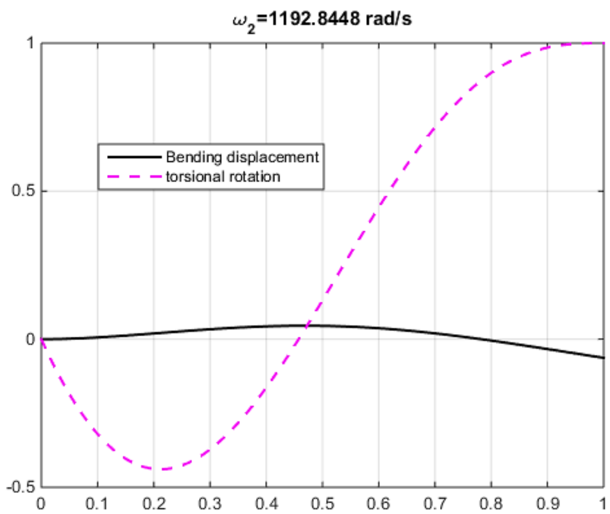
Frequency number	Timoshenko beam model [rad/s]	Reference [13] [rad/s]
1	193,19	193,2
2	1192.84	1192,9
3	3262.37	3262,2
4	4073,17	4073,2

2. **Mode Shapes**

The mode shapes of the composite thick Timoshenko beam ($S = 0.03528$) obtained by the system (2) are shown in figures 4(a), 4(b), 4(c) and 4(d).

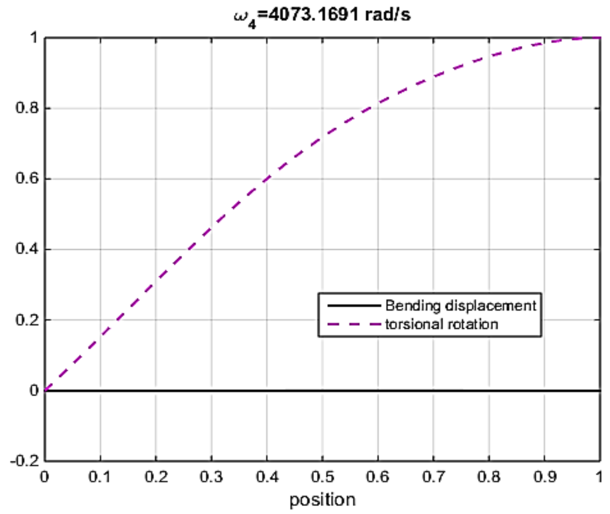


(a)

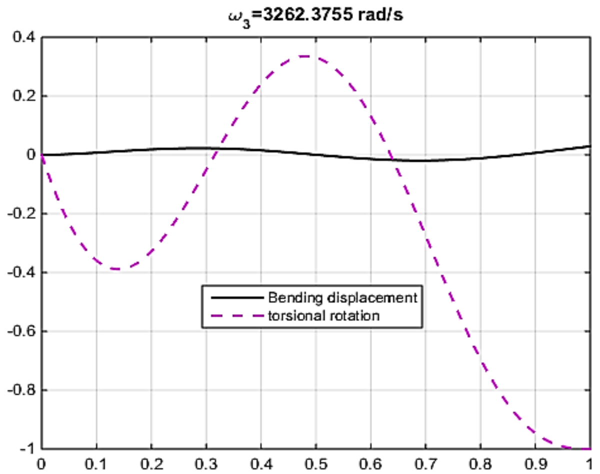


(b)

Figure 4. (a), (b) Mode shapes and natural frequencies of composite Timoshenko single composite layer $\theta = 15°$



(c)



(d)

Figure 4. (c), (d) Mode shapes and natural frequencies of composite Timoshenko single composite layer $\theta = 15°$

The obtained mode shapes results using the Timoshenko beam model of cantilever unidirectional composite beam $\theta = 15°$ are very similar with the results obtained by Banerjee [1] and Banerjee and Williams [3] using DSM approach.

4.2.2 The natural frequencies for various ply orientation angle

Varying the ply orientation angle from 0° to 90°, the five natural frequencies obtained by modal analysis are shown in figures 5 and 6 by using the Euler-Bernoulli and the Timoshenko finite element beam respectively.

The figures below show clearly that the curves of each frequency are very similar for both models (Euler-Bernoulli and Timoshenko). It is remarkable that the frequencies obtained by the Timoshenko beam model are less than the results obtained by the Euler-Beam model for all ply orientation angles.

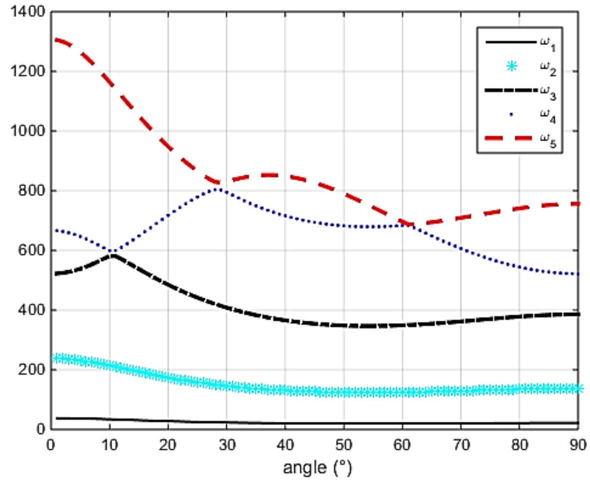


Figure 5. The first five natural frequencies for various angle-ply orientation obtained by the Euler-Bernoulli beam

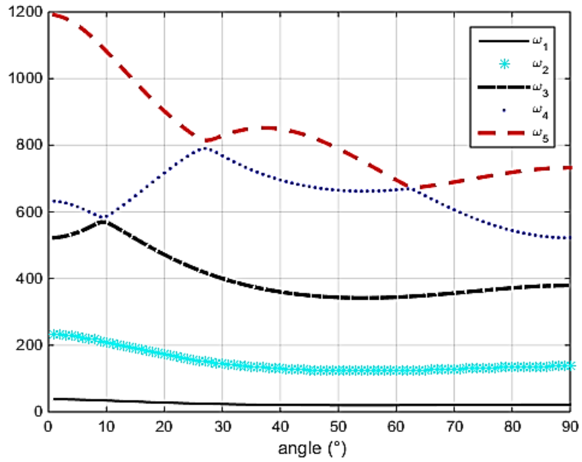


Figure 6. The first five natural frequencies for various angle-ply orientation obtained by the Timoshenko beam $(S = 0.03528)$.

4.3 The effects of shear on natural frequencies

The effects of shear deformation on the three first frequencies of the single layer composite beam $[\theta = 15°]$obtained by Finite Elements Analysis program is presented in this paragraph. Figure 7 shows the three first natural frequencies for different values of shear deformation compared to the results obtained from Euler-Bernoulli results.

$\boldsymbol{\omega_0}$ Presents the frequencies results obtained by Euler-Bernoulli beam theory $(S = 0)$ (see table 1). It appears clearly that the results obtained by varying the shear deformation from 0 to 0.1 the non-dimensional natural frequencies decrease for the three first natural frequencies, where the difference between the first and third curves is remarkable. The obtained results are very similar to J. R Banerjee [1] and Teoh and Huang results [2].

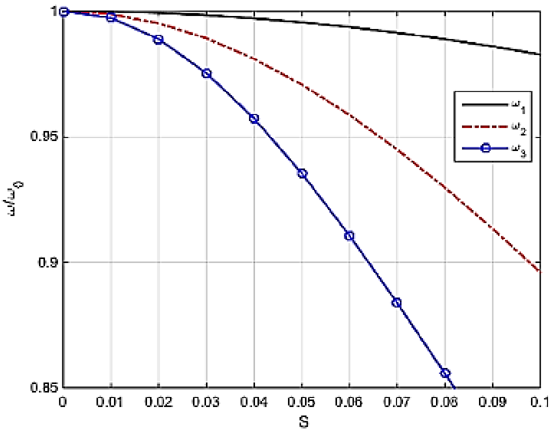


Figure 7. Shear deformation effects on natural frequencies of thick beam

4.4 Frequency Response Analysis

A spectrum response analysis is investigated, in this part. The FFT result at the tip of the clamped-free beam using an impulse force is presented for different shear deformation values S: 0.02, 0.04, 0.06, 0.08 and 0.1. Figures 8(a), 8(b) and 8(c) show the FFT result of the single ply layer $\theta = 15°$ for the different shear values. The results show that there are three peaks of resonance in the 700 first frequencies range; each peak corresponds to the first, second and third frequencies.

- It is observed that the absolute values of the FFT result increase with the variation of the shear deformation values for the three first frequencies.
- Figure 8(a) shows that the 1st resonance frequency is almost similar for the different values of shear deformation.
- Figure 8(b) shows a slight difference in the 2nd resonance frequency for the different values of shear deformation.
- Figure 8(c) shows a remarkable difference in the 3rd resonance frequency for the different values of shear deformation.

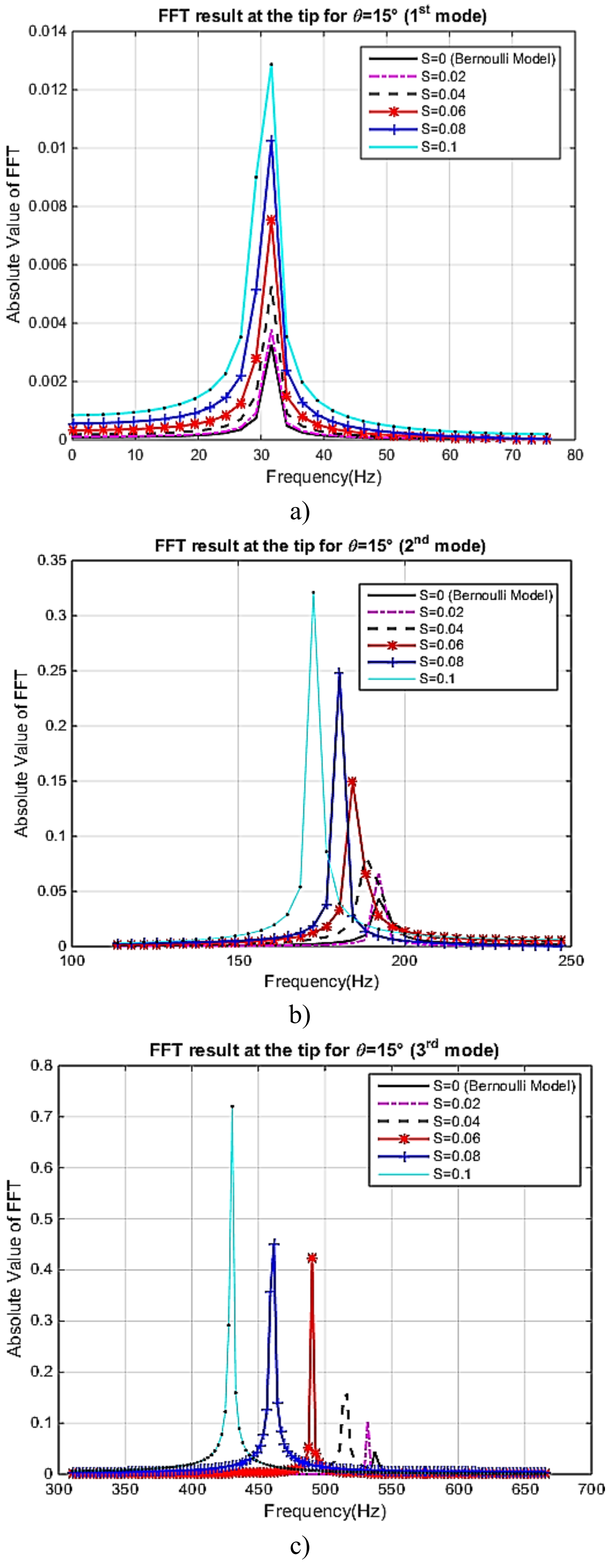


Figure 8. The FFT result of the clamped-free beam with a single layer $\theta = 15°$.

5. CONCLUSION

A finite element method is developed to illustrate the modal analysis and frequency response analysis of a clamped-free single layer ply orientation angle composite beam using Euler-Bernoulli and Timoshenko beam approximations, in this paper.

The results obtained by free vibration analysis using the Timoshenko beam model are validated with the dynamic stiffness method of unidirectional composite layer beam $\theta = 15°$. The mode shapes and their associate frequencies results obtained from the Finite Element Timoshenko beam model are very similar with those obtained by using the dynamic stiffness method DSM.

The obtained three first dimensionless frequencies from modal analysis curves show that the variation of the transverse shear deformation has influence in natural frequencies.

The Absolute values of FFT (Fast Fourier Transform) response result of an impulse force excitation for various shear deformation values is presented using a MATLAB program.

The FFT result at the tip for the beam for various shear deformation 0.02, 0.04, 0.06, 0.08 and 0.1 demonstrate the frequency response of the cantilever beam. These results explain the influence of the transverse shear deformation on the frequencies of resonance of the composite beam, where the 1st frequency is similar varying the shear deformation values, a difference between the frequencies of resonance (2nd and 3rd frequencies) becomes remarkable and decrease with the increasing of the transverse shear deformation. The Absolute value of FFT increase also with the increasing of shear deformation.

The frequency response results obtained from the finite element approximation by using the Timoshenko and Euler-Bernoulli beam models for the unidirectional composite beam $\theta = 15°$ give an important idea for the unidirectional ply orientation angle section in the first time, and it can be generalized for laminated thick beams.

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