Free Vibration Analysis of Anti-Symmetric FGM Sandwich Circular Beams Using a Fifth-Order Circular Beam Theory

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Abstract: - This paper focused on the fundamental frequency analysis of anti-symmetric FGM sandwich circular beams. A fifth-order circular beam theory considering the influence of transverse shear and normal strains is developed in this study. Fifth-order terms in terms of thickness coordinates are considered for the first time in the displacement field of the beam to solve the free vibration problems of circular beams. The theory assumes fifth-order variations of thickness coordinates in axial (tangential) displacement and fourth-order variations of thickness coordinates in transverse (radial) displacements. Hamilton's principle is applied to derive equations of motion. An exact solution for the frequency analysis of a simply-supported circular beam is obtained using the Navier technique. Anti-symmetric FGM sandwich circular beams are considered for numerical studies. The material properties of face sheets are graded in the thickness direction of the beam according to the power law whereas the core of the beam is made up of isotropic material. The fundamental frequencies obtained for different values of radius of curvature (R), the power-law index (p) and lamination schemes of S-P-FGM sandwich circular beams are presented for the first time in this study and can be considered as the main contribution of this study. For the verification of the present theory, fundamental frequencies for straight beams obtained using the present theory are compared with the previously published papers and found in good agreement with those. Based on the comparison of the numerical results and discussion it is concluded that for the same length and thickness, the value of non-dimensional fundamental frequency increases as the radius of curvature is decreased. Also, the non-dimensional frequency decreases as the power-law index increases.

Keywords: - Fundamental frequencies, circular beam, anti-symmetric, functionally graded sandwich, fifth-order circular beam theory, transverse normal strain.

1. INTRODUCTION

Circular beams made up of functionally graded advanced composite materials are widely used in many industrial structures. Functionally graded material (FGM) is a combination of two materials in which their properties are continuously varying in single or multiple directions according to the different gradation rules. The power law is the most popularly used rule for the gradation of material properties. Circular beams are often subjected to dynamic forces or conditions; therefore, vibration analysis is one of the most important aspects of the design of FGM sandwich circular beams.

Beam structures made up of homogenous or composite materials are analyzed using the theory of elasticity. But, the elasticity solution for the free vibration analysis of circular beams is difficult due to curvature effects. Therefore, researchers have developed approximate beam theories which give approximate solutions for the vibration problem of a circular beam with good accuracy compared to the theory of elasticity. The classical beam theory (CBT) developed by Bernoulli-Euler [1] gives more accurate results for slender beams. Since the CBT does not consider the effects of shear deformation, it overestimates the fundamental frequencies of the thick circular beams.

This limitation of CBT is overcome by the Timoshenko beam theory [2] (TBT). The theory is also known as first-order shear deformation theory (FSDT). The TBT is the first approximate theory that considers the effects of transverse shear deformation and has given improved results over the CBT. However, the TBT is also not accurate in predicting the global response of the thick circular beams because this theory requires problem-dependent shear correction factors to account for energy due to shear. Also, this theory shows constant transverse shear stress across the thickness of the beam which is not realistic. This is the reason why researchers have paid attention to the development of refined beam theories which consider the effects of transverse shear and normal deformations along with the rotary inertia. These theories are systematically documented by Sayyad and Ghugal [3, 4]. Some good studies related to functionally graded circular beams are found in the literature. Malekzadeh et al. [5]-[6] presented an out-of-plane free vibration analysis of FGM circular beams using the differential quadrature method based on the first-order shear deformation theory. Alshorbagy et al. [7] presented a free vibration analysis of bi-directional FGM beams using the finite element method in conjunction with Euler–Bernoulli beam theory. Yousefi and Rastgoo [8] presented the free vibration analysis of functionally graded spatial curved beams based on the first-order shear deformation theory and the Ritz method. The material properties are graded in the direction of the curvature of the curved beam. Piovan et al. [9] have developed a model of non-homogeneous and/or FGM curved beams. The finite element method is used to discretize the motion equations to solve problems of dynamics, statics, and buckling. Arefi [10] has presented an elastic solution of a curved beam made of functionally graded materials with different cross sections. Kurtaran [11] presented large displacement static and transient analysis of moderately thick deep functionally graded curved beams with constant curvature using the generalized differential quadrature method. Tufekci et al. [12] employed Eringen's nonlocal elasticity theory with a classical beam model considering the effects of axial extension and the shear deformation for the static analysis of the nonlocal curved beams. Pydah and Sabale [13, 14]; Pydah and Batra [15]; Fariborz and Batra [16] have presented static and vibration analysis of bi-directional functionally graded circular beams. Huynh et al. [17] have presented bending, buckling, and free vibration analyses of functionally graded curved beams with variable curvatures using an isogeometric approach based on the Timoshenko beam theory. Similarly, Ebrahimi and Daman [18] have also applied the Timoshenko beam theory for the analysis of curved nanobeam.

He et al. [19] have developed analytical solutions for the bending analysis of functionally graded curved beams with different properties in tension and compression.

Zhao et al. [20] have presented free vibration analyses of moderately thick functionally graded porous curved beams using the first-order beam theory. She et al. [21] studied the resonance behavior of porous FG curved nanobeams. Wan et al. [22] applied the first-order beam theory for the geometrically nonlinear analysis of an FGM curved beam with variable curvature. Pandey and Pradyumna [23] have presented a thermal shock analysis of functionally graded sandwich curved beams using a layerwise theory. Beg and Yasin [24] presented bending, free, and forced vibration of functionally graded curved beams in the thermal environment using a layerwise theory. Beg et al. [25] performed the static and free vibration analysis of porous FGM curved beams using third-order beam theory. Nikrad et al. [26] have presented a large deformation analysis of FGM porous curved beams in a thermal environment. The first-order shear deformation theory along with the Rayleigh–Ritz method and the Newton–Raphson method is used for the analysis. Draiche et al. [27] have developed an integral shear and normal deformation theory for bending analysis of functionally graded sandwich curved beams. Belarbi et al. [28]-[29] have developed a finite element formulation for the bending, buckling, and free vibration analysis of functionally graded sandwich curved beams via a higher-order shear deformation theory. Vlase et al. [30] presented a semi-analytical method to simplify the calculus of the eigenmodes of a mechanical system with bars. The method is applied to symmetrical structures.

1.1. The shortcomings of the literature review

The authors have reviewed plenty of research papers on bending, buckling, and free vibration analysis of FGM straight as well as circular beams. Based on the literature reviewed, the following observations are made.

1. The authors have found plenty of research papers on static and free vibration analysis of straight FGM sandwich beams using higher-order refined beam theories such as Reddy [31], [32], Simsek [32], Thai and Vo [33], Vo et al. [34]-[36], Nguyen et al. [37]-[38], Sayyad and Ghugal [39] -[40], Sayyad and Avhad [41], etc. These are selected papers on the vibration analysis of straight FGM sandwich beams. There are many more research papers available on straight FGM sandwich beams [4]. However, many refined

theories presented in the past literature do not consider the influence of transverse normal strains on the frequency analysis of FGM sandwich beams.
2. The authors have found limited literature on the free vibration analysis of single-layer FGM circular beams. However, the authors do not find any research paper on the free vibration analysis of S-P-FGM (sigmoid) and anti-symmetric FGM sandwich circular beams.
3. In the previous research, researchers have used a third-order expansion of thickness coordinates in the displacement field of the refined beam theory. However, expansion of thickness coordinates up to the fifth-order in the axial displacement (transverse shear strain) and up to the fourth-order in the transverse displacement (transverse normal strain) is not found in the literature.

This motivates the authors to carry out free vibration analysis of anti-symmetric S-P-FGM and sandwich circular beams. The novelty of the present study is highlighted in the next section.

1.2. The novelty of the present work

1) A fifth-order curved beam theory considering the influence of transverse shear and normal strains is developed by Avhad and Sayyad [42, 43] for the static and free vibration analysis of laminated composites, sandwich, and functionally graded circular beams. In this article, this theory is applied to the free vibration analysis of anti-symmetric FGM sandwich circular beams.
2) A fifth-order circular beam theory is applied for the first time to solve the free vibration problems of Sigmoid FGM circular beams.
3) The present theory considers the effects of both transverse shear and normal strains.
4) The frequency results of functionally graded sigmoid circular beams can be serving as the benchmark for future researchers.

The equations of motion of the present theory are derived using Hamilton's principle. Exact analytical solutions to free vibration problems are obtained using Navier's technique. Fundamental frequencies are obtained for two-layered sigmoid (S-P-FGM) and three-layered anti-symmetric FGM sandwich (2-2-1, 2-1-1) circular beams. Effects of the power-law index and the radius of curvature on the fundamental frequencies of circular beams are investigated.

1.3. The geometry of the circular beam

A simply-supported FGM sandwich circular beam is considered in this study. The circular beam has a radius of curvature (R), curved length (L) in the x-direction, unit width (b = 1) in the y-direction, and thickness (h) in the z-direction. These geometrical details of the circular beam are shown in Fig. 1. A sandwich circular beam consists of three layers made up of functionally graded material whose thickness is distributed among three layers anti-symmetrically. The top and bottom layers (face sheets) of the circular beam are made up of functionally graded materials whereas the middle layer (the core) is made up of isotropic material (ceramic).

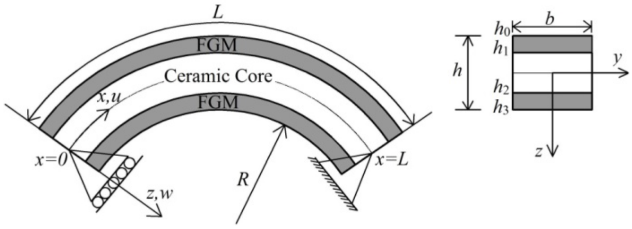

Figure. 1 Displacement parameters and geometry details of the FGM sandwich circular beam

1.4. Material properties of the circular beam

Two types of FGM circular beams are analyzed in the present study. The material properties (E, G, μ, ρ) of these two types of circular beams are graded across the thickness using the power law stated in Eq. (1).

$$E(z) = E_m + (E_c - E_m)V_c(z)$$
$$\rho(z) = \rho_m + (\rho_c - \rho_m)V_c(z) \tag{1}$$

where $E(z)$ is the modulus of elasticity, $\rho(z)$ is the density of the material, and V_c (z) is the volume fraction function. Subscripts m and c correspond to metal and ceramic constituents, and p is the power-law index. Values of the volume fraction function for different layers of the FGM sandwich circular beams under consideration are given below.

Type A: Three-layered anti-symmetric FGM sandwich circular beams

Values of the volume fraction function for the three-layered FGM sandwich circular beam are given by Eq. (2).

$$\text{Layer 1:} \quad V_c(z) = \left(\frac{z-h_0}{h_1-h_0}\right)^p \quad \text{for} \quad z \in [h_0,\ h_1]$$
$$\text{Layer 2:} \quad V_c(z) = 1 \quad \text{for} \quad z \in [h_1,\ h_2] \tag{2}$$
$$\text{Layer 3:} \quad V_c(z) = \left(\frac{z-h_3}{h_2-h_3}\right)^p \quad \text{for} \quad z \in [h_2,\ h_3]$$

Through-the-thickness distributions of modulus of elasticity (E) for three-layered FGM sandwich circular beams are shown in Fig. 2.

Type B: Two-layered sigmoid (S-P-FGM) circular beams

The sigmoid law for the gradation of material properties is specifically used for two-layered beams. The sigmoid law is a combination of two power-law functions (S-P-FGM). Eq. (3) shows the values of the volume fraction function for the two-layered anti-symmetric (S-P-FGM) circular beams.

$$\begin{aligned} &\text{Layer1:} \quad V_c(z) = 1 + \left(\frac{z}{h} - \frac{1}{2}\right)^p \quad z \in [-h/2,\ 0] \\ &\text{Layer2:} \quad V_c(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^p \quad z \in [0,\ h/2] \end{aligned} \tag{3}$$

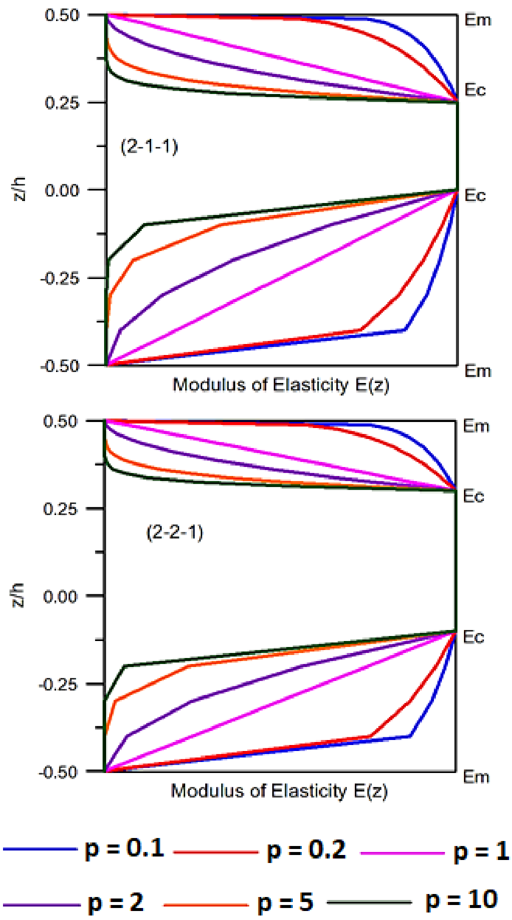

Figure. 2 Distributions of the modulus of elasticity through the thickness of three-layered anti-symmetric FGM sandwich circular beam

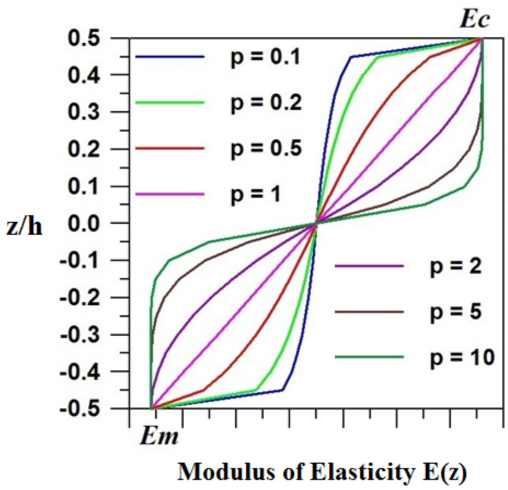

Figure 3. Distributions of the modulus of elasticity through the thickness of two-layered anti-symmetric S-P-FGM circular beam

Through-the-thickness distributions of modulus of elasticity (E) for two-layered anti-symmetric S-P-FGM circular beams are shown in Fig. 3. However, distributions of other material properties (G, μ, ρ) through the thickness of the beam follow the same trend.

2 A FIFTH-ORDER CIRCULAR BEAM THEORY

A fifth-order circular beam theory considering the influence of transverse shear strain (γ_{xz}) and transverse normal strains (ε_z) is developed in this study. The theory assumes fifth-order variation of thickness coordinates in axial (tangential) displacement (w) and fourth-order variations of thickness coordinates in transverse (radial) displacements. The x-directional (axial) displacement (u) consists of the extension, bending, and shearing components whereas the z-directional (transverse) displacement is a function of both x and z coordinates accounting for the effects of transverse normal strain. The displacement field of the present fifth-order circular beam theory at an arbitrary material point within the circular beam is written as per Eq. (4):

$$\begin{aligned} u(x,z,t) &= F_0\, u_0(x,t) + F_1\, u_1(x,t) \\ &\quad + F_2\, u_2(x,t) + F_3\, u_3(x,t) \\ w(x,z,t) &= w_0(x,t) + \frac{dF_2}{dz} w_1(x,t) \\ &\quad + \frac{dF_3}{dz} w_2(x,t) \end{aligned} \tag{4}$$

where

$$\begin{aligned} &F_0 = \left(1 + \frac{z}{R}\right), \quad F_1 = -z, \quad F_2 = \left(z - \frac{4z^3}{3h^2}\right), \\ &F_3 = \left(z - \frac{16z^5}{5h^4}\right), \quad u_1 = \frac{\partial w_0}{\partial x}, \quad u_2 = \phi_x, \\ &u_3 = \psi_x, \quad w_1 = \phi_z, \quad w_2 = \psi_z \end{aligned} \tag{5}$$

where u and w are the x- (tangential) and z- (radial) directional displacements of an arbitrary point of the circular beam domain; u_0 and w_0 are the x- directional and z- directional displacements of a point on the neutral axis of the circular beam; $1 / (1 + z/R)$ is the Lame's parameter, and $(\phi_x, \psi_x, \phi_z, \psi_z)$ are the rotations of a cross-section of the beam. The displacement field of transverse displacement shows that $w = w_0$ at $z = \pm h/2$ and varies through the thickness of the circular beam. Lame's parameter is neglected for the strain calculation of circular beams. Eq. (6) shows the non-zero normal and transverse shear strains obtained at any point of the circular beam using the theory of elasticity.

$$\begin{aligned}\varepsilon_x &= \frac{\partial u_0}{\partial x} + F_1\frac{\partial^2 w_0}{\partial x^2} + F_2\frac{\partial \phi_x}{\partial x} + F_3\frac{\partial \psi_x}{\partial x} + \frac{w_0}{R}\\ &\quad + \frac{dF_2}{dz}\frac{\phi_z}{R} + \frac{dF_3}{dz}\frac{\psi_z}{R}\\ \varepsilon_z &= \frac{d^2F_2}{dz^2}\phi_z + \frac{d^2F_3}{dz^2}\psi_z\\ \gamma_{xz} &= \frac{dF_2}{dz}\phi_x + \frac{dF_3}{dz}\psi_x + \frac{dF_2}{dz}\frac{\partial \phi_z}{\partial x} + \frac{dF_3}{dz}\frac{\partial \psi_z}{\partial x}\end{aligned} \tag{6}$$

where

$$\begin{aligned}\frac{dF_2}{dz} &= \left(1-\frac{4z^2}{h^2}\right), \quad \frac{dF_3}{dz} = \left(1-\frac{16z^4}{h^4}\right)\\ \frac{d^2F_2}{dz^2} &= \left(-\frac{8z}{h^2}\right), \quad \frac{d^2F_3}{dz^2} = \left(-\frac{64z^3}{h^4}\right)\end{aligned} \tag{7}$$

Eq. (6) shows that the transverse shear strain is zero at $z = \pm h/2$ and the present theory satisfies the traction-free boundary conditions at the top and bottom surfaces of the beam. The two-dimensional constitutive relation stated in Eq. (8) is used to obtain axial and transverse shear stresses at the n^{th} layer (n = 1, 2, 3) of the sandwich circular beam.

$$\begin{Bmatrix}\sigma_x\\ \sigma_z\\ \tau_{xz}\end{Bmatrix}^n = \begin{bmatrix}Q_{11}(z) & Q_{13}(z) & 0\\ Q_{13}(z) & Q_{33}(z) & 0\\ 0 & 0 & Q_{55}(z)\end{bmatrix}^n \begin{Bmatrix}\varepsilon_x\\ \varepsilon_z\\ \gamma_{xz}\end{Bmatrix}^n \tag{8}$$

where $Q_{ij}(z)$ are the stiffness coefficients in terms of engineering constants and are defined in the Eq. (9).

$$\begin{aligned}Q_{11}(z) = Q_{33}(z) = \frac{E(z)}{1-\mu^2},\ Q_{13}(z) = \frac{\mu E(z)}{1-\mu^2},\\ Q_{55}(z) = \frac{E(z)}{2(1+\mu)}\end{aligned} \tag{9}$$

Hamilton's principle stated in Eq. (10) is used to derive equations of motion associated with the present theory.

$$\begin{aligned}&\int_0^L \int_{-h/2}^{h/2} (\sigma_x \delta\varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})\ dz\, dx\\ &+ \int_0^L \int_{-h/2}^{h/2} \rho(z)\left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 w}{\partial t^2}\delta w\right) dz\, dx\\ &- \int_0^L q\ \delta w\, dx = 0\end{aligned} \tag{10}$$

where δ denotes the variational operator. Equations of motion are derived after integrating Eq. (10) by parts, collecting the coefficients of unknown variables, and setting them equal to zero ($\delta u_0 = 0$, $\delta w_0 = 0$, $\delta\phi_x = 0$, $\delta\psi_x = 0$, $\delta\phi_z = 0$, $\delta\psi_z = 0$). Eqs. (11) through (16) are the equations of motion associated with the present theory.

δu_0:

$$\begin{aligned}&\frac{\partial N_x}{\partial x} + \left(I_1 + 2\frac{I_2}{R} + \frac{I_3}{R^2}\right)\frac{\partial^2 u_0}{\partial t^2} - \left(I_2 + \frac{I_3}{R}\right)\frac{\partial^3 w_0}{\partial t^2 \partial x}\\ &+ \left(I_4 + \frac{I_5}{R}\right)\frac{\partial^2 \phi_x}{\partial t^2} + \left(I_6 + \frac{I_7}{R}\right)\frac{\partial^2 \psi_x}{\partial t^2} = 0\end{aligned} \tag{11}$$

δw_0:

$$\begin{aligned}&\frac{\partial^2 M_x^b}{\partial x^2} - \frac{N_x}{R} + q - \left(I_2 + \frac{I_3}{R}\right)\frac{\partial^3 u_0}{\partial t^2 \partial x} - I_3\frac{\partial^4 w_0}{\partial t^2 \partial x^2}\\ &+ I_5\frac{\partial^3 \phi_x}{\partial t^2 \partial x} + I_7\frac{\partial^3 \psi_x}{\partial t^2 \partial x} + I_1\frac{\partial^2 w_0}{\partial t^2} + I_{11}\frac{\partial^2 \phi_z}{\partial t^2}\\ &+ I_{12}\frac{\partial^2 \psi_z}{\partial t^2} = 0\end{aligned} \tag{12}$$

$\delta\phi_x$:

$$\begin{aligned}&\frac{\partial M_x^{s1}}{\partial x} - Q_{xz}^1 + \left(I_4 + \frac{I_5}{R}\right)\frac{\partial^2 u_0}{\partial t^2} - I_5\frac{\partial^3 w_0}{\partial t^2 \partial x}\\ &+ I_8\frac{\partial^2 \phi_x}{\partial t^2} + I_9\frac{\partial^2 \psi_x}{\partial t^2} = 0\end{aligned} \tag{13}$$

$\delta\psi_x$:

$$\begin{aligned}&\frac{\partial M_x^{s2}}{\partial x} - Q_{xz}^2 + \left(I_6 + \frac{I_7}{R}\right)\frac{\partial^2 u_0}{\partial t^2} - I_7\frac{\partial^3 w_0}{\partial t^2 \partial x}\\ &+ I_9\frac{\partial^2 \phi_x}{\partial t^2} + I_{10}\frac{\partial^2 \psi_x}{\partial t^2} = 0\end{aligned} \tag{14}$$

$\delta\phi_z$:

$$\begin{aligned}&\frac{\partial Q_{xz}^1}{\partial x} - \frac{V_x^1}{R} - Q_z^1 + q f_1'(z) + I_{11}\frac{\partial^2 w_0}{\partial t^2}\\ &+ I_{13}\frac{\partial^2 \phi_z}{\partial t^2} + I_{14}\frac{\partial^2 \psi_z}{\partial t^2} = 0\end{aligned} \tag{15}$$

$\delta\psi_z$:

$$\begin{aligned}&\frac{\partial Q_{xz}^2}{\partial x} - \frac{V_x^2}{R} - Q_z^2 + q f_2'(z) + I_{12}\frac{\partial^2 w_0}{\partial t^2}\\ &+ I_{14}\frac{\partial^2 \phi_z}{\partial t^2} + I_{15}\frac{\partial^2 \psi_z}{\partial t^2} = 0\end{aligned} \tag{16}$$

The boundary conditions at supports $x = 0$ and $x = L$ are of the following form.

Either $N_x = 0$ or $u_0 = 0$; $M_x^b = 0$ or $\frac{\partial w_0}{\partial x} = 0$; $\frac{\partial M_x^b}{\partial x} = 0$ or $w_0 = 0$; $M_x^{s1} = 0$ or $\phi_x = 0$; $M_x^{s2} = 0$ or $\psi_x = 0$; $Q_{xz}^1 = 0$or $\phi_z = 0$ and $Q_{xz}^2 = 0$or $\psi_z = 0$

where axial force, bending moments, and shear force resultants of the circular beam are defined as Eq. (17).

$$\begin{aligned}&(N_x,\ M_x^b,\ M_x^{s1},\ M_x^{s2}) = \int_{-h/2}^{h/2} \sigma_x\ (1,\ z,\ F_2,\ F_3) dz,\\ &(V_x^1,\ V_x^2) = \int_{-h/2}^{h/2} \sigma_x \left(\frac{dF_2}{dz},\ \frac{dF_3}{dz}\right)\ dz,\\ &(Q_z^1,\ Q_z^2) = \int_{-h/2}^{h/2} \sigma_z \left(\frac{d^2F_2}{dz^2},\ \frac{d^2F_3}{dz^2}\right)\ dz,\\ &(Q_{xz}^1,\ Q_{xz}^2) = \int_{-h/2}^{h/2} \tau_{xz} \left(\frac{dF_2}{dz},\ \frac{dF_3}{dz}\right)\ dz\end{aligned} \tag{17}$$

Inertia constants appearing in the set of equations of motion are defined in Eq. (18).

$$(I_1, I_2, I_3, I_4, I_5, I_6, I_7) = \int_{-h/2}^{h/2} \rho(z)[1, z, z^2, F_2, F_2\, z, F_3, F_3\, z] dz$$

$$(I_8, I_9, I_{10}, I_{11}, I_{12}) = \int_{-h/2}^{h/2} \rho(z) \left[F_2{}^2, F_2 F_3, F_3{}^2, \frac{dF_2}{dz}, \frac{dF_3}{dz}\right] dz \tag{18}$$

$$(I_{13}, I_{14}, I_{15}) = \int_{-h/2}^{h/2} \rho(z) \left\{\left(\frac{dF_2}{dz}\right)^2, \left(\frac{dF_2}{dz}\frac{dF_3}{dz}\right), \left(\frac{dF_3}{dz}\right)^2\right\} dz$$

3 AN EXACT ANALYTICAL SOLUTION

In this section, the Navier solution technique is used to obtain an exact analytical solution for the free vibration analysis of simply supported anti-symmetric FGM sandwich circular beams. Eq. (19) defines the boundary conditions at the simply-supported edges.

$$\begin{gathered} N_x = 0,\ w_0 = 0, \\ M_x^b = 0,\ M_x^{s1} = 0, \\ M_x^{s2} = 0, \\ \phi_z = 0,\ \psi_z = 0 \end{gathered} \tag{19}$$

Unknown variables involved in the equations of motion are assumed in the following form stated in Eq. (20) to satisfy the simply-supported boundary conditions stated in Eq. (19) exactly.

$$\begin{gathered} \begin{Bmatrix} u_0 \\ \phi_x \\ \psi_x \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} u_m \\ \phi_{xm} \\ \psi_{xm} \end{Bmatrix} \cos(\alpha x)\, sin(\omega t) \\ \begin{Bmatrix} w_0 \\ \phi_z \\ \psi_z \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} w_m \\ \phi_{zm} \\ \psi_{zm} \end{Bmatrix} sin(\alpha x)\, sin(\omega t) \end{gathered} \tag{20}$$

where

$\alpha = m\pi/L$, ω is the natural frequency, u_m, w_m, ϕ_{xm}, ψ_{xm}, ϕ_{zm} and ψ_{zm} are the amplitudes.

Substitution of Eq. (20) into equations of motion (11) through (16) leads to the Eigenvalue problem stated in Eq. (21).

$$\{[K]_{6\times 6} - \omega^2 [M]_{6\times 6}\} \times \{\Delta\} = 0 \tag{21}$$

where elements of stiffness matrix [K] are defined in the Eqs. (22) through (26).

$$\begin{gathered} K_{11} = A_{11}\alpha^2, K_{12} = \left(-\frac{A_{11}}{R}\alpha - B_{11}\alpha^3\right), \\ K_{13} = C_{11}\alpha^2, \\ K_{14} = (D_{11}\alpha^2), K_{15} = \left(-\frac{E_{11}}{R}\alpha - G_{13}\alpha\right), \\ K_{16} = -\frac{F_{11}}{R}\alpha - H_{13}\alpha, \\ K_{22} = \left(\frac{A_{11}}{R^2} + I_{11}\alpha^4 + 2\frac{B_{11}}{R}\alpha^2\right), \\ K_{23} = \left(-\frac{C_{11}}{R}\alpha - J_{11}\alpha^3\right), \\ K_{24} = \left(-\frac{D_{11}}{R}\alpha - K_{11}\alpha^3\right), \\ K_{25} = \left(\frac{E_{11}}{R^2} + \frac{G_{13}}{R} + \frac{L_{11}}{R}\alpha^2 + N_{13}\alpha^2\right), \\ K_{26} = \left(\frac{F_{11}}{R^2} + \frac{H_{13}}{R} + \frac{M_{11}}{R}\alpha^2 + O_{13}\alpha^2\right), \\ K_{33} = (P_{11}\alpha^2 + BK_{55}), \\ K_{34} = (QQ_{11}\alpha^2 + BL_{55}), \\ K_{35} = \left(-\frac{R_{11}}{R}\alpha - T_{13}\alpha + BK_{55}\alpha\right), \\ K_{36} = \left(-\frac{S_{11}}{R}\alpha - U_{13}\alpha + BL_{55}\alpha\right), \\ K_{44} = (V_{11}\alpha^2 + BM_{55}), \\ K_{45} = \left(-\frac{W_{11}}{R}\alpha - Y_{13}\alpha + BL_{55}\alpha\right), \\ K_{46} = \left(-\frac{X_{11}}{R}\alpha - Z_{13}\alpha + BM_{55}\alpha\right), \\ K_{55} = \left(\frac{BA_{11}}{R} + 2\frac{BC_{13}}{R} + BH_{33} + BK_{55}\alpha^2\right), \\ K_{56} = \left(\frac{BB_{11}}{R} + \frac{BD_{13}}{R} + \frac{BF_{13}}{R} + BI_{33} + BL_{55}\alpha^2\right), \\ K_{66} = \left(\frac{BE_{11}}{R} + 2\frac{BG_{13}}{R} + BJ_{33} + BM_{55}\alpha^2\right). \end{gathered} \tag{22}$$

where

$$\begin{gathered} (A_{11}, B_{11}, C_{11}, D_{11}, E_{11}, F_{11}) = \int_{-h/2}^{h/2} Q_{11}(z) \left[1, z, F_2, F_3, \frac{dF_2}{dz}, \frac{dF_3}{dz}\right]\ dz, \\ (II_{11}, J_{11}, K_{11}, L_{11}, M_{11}) = \int_{-h/2}^{h/2} Q_{11}(z)\ z\left[z, F_2, F_3, \frac{dF_2}{dz}, \frac{dF_3}{dz}\right]\ dz, \\ (G_{13}, H_{13}) = \int_{-h/2}^{h/2} \left[\frac{d^2F_2}{dz^2}, \frac{d^2F_3}{dz^2}\right]\ dz, \end{gathered} \tag{23}$$

$$\begin{gathered} (N_{13}, O_{13}) = \int_{-h/2}^{h/2} Q_{13}(z) z \left[\frac{d^2F_2}{dz^2}, \frac{d^2F_3}{dz^2}\right]\ dz, \\ (P_{11}, QQ_{11}, R_{11}, S_{11}) = \int_{-h/2}^{h/2} Q_{11}(z) F_2 \left[F_2, F_3, \frac{dF_2}{dz}, \frac{dF_3}{dz}\right]\ dz, \\ (T_{13}, U_{13}) = \int_{-h/2}^{h/2} Q_{13}(z) F_2 \left[\frac{d^2F_2}{dz^2}, \frac{d^2F_3}{dz^2}\right]\ dz, \\ (V_{11}, W_{11}, X_{11}) = \int_{-h/2}^{h/2} Q_{11}(z) F_3 \left[F_3, \frac{dF_2}{dz}, \frac{dF_3}{dz}\right]\ dz, \end{gathered} \tag{24}$$

$$(BA_{11}, BB_{11}) = \int_{-h/2}^{h/2} Q_{11}(z) \frac{dF_2}{dz} \left[\frac{dF_2}{dz}, \frac{dF_3}{dz}\right] dz,$$
$$(Y_{13}, Z_{13}) = \int_{-h/2}^{h/2} Q_{13}(z) F_2 \left[\frac{d^2F_2}{dz^2}, \frac{d^2F_3}{dz^2}\right] dz,$$
$$(BC_{13}, BD_{13}) = \int_{-h/2}^{h/2} Q_{13}(z) \frac{dF_2}{dz} \left[\frac{d^2F_2}{dz^2}, \frac{d^2F_3}{dz^2}\right] dz, \quad (25)$$
$$(BF_{13}, BG_{13}) = \int_{-h/2}^{h/2} Q_{13}(z) \frac{dF_3}{dz} \left[\frac{d^2F_2}{dz^2}, \frac{d^2F_3}{dz^2}\right] dz,$$

$$(BH_{33}, BI_{33}) = \int_{-h/2}^{h/2} Q_{33}(z) \frac{d^2F_2}{dz^2} \left[\frac{d^2F_2}{dz^2}, \frac{d^2F_3}{dz^2}\right] dz,$$
$$(BK_{55}, BL_{55}) = \int_{-h/2}^{h/2} Q_{55}(z) \frac{dF_2}{dz} [f_1'(z), f_2'(z)]\, dz,$$
$$BJ_{33} = \int_{-h/2}^{h/2} Q_{33}(z) \left[\frac{d^2F_3}{dz^2} \frac{d^2F_3}{dz^2}\right] dz, \quad (26)$$
$$BE_{11} = \int_{-h/2}^{h/2} Q_{11}(z) \left[\frac{dF_2}{dz} \frac{dF_2}{dz}\right] dz,$$
$$BM_{55} = \int_{-h/2}^{h/2} Q_{55}(z) \left[\frac{dF_3}{dz} \frac{dF_3}{dz}\right] dz$$

Elements of mass matrix [M] are defined in Eq. (27).

$$\begin{gathered} M_{11} = \left(I_1 + 2\frac{I_2}{R} + \frac{I_3}{R^2}\right), M_{12} = -\left(I_2 + \frac{I_3}{R}\right)\alpha, \\ M_{13} = \left(I_4 + \frac{I_5}{R}\right), M_{14} = \left(I_6 + \frac{I_7}{R}\right), M_{15} = 0, \\ M_{16} = 0,\ M_{22} = (I_3\alpha^2 + I_1),\ M_{23} = -I_5, \\ M_{24} = -I_7\alpha,\ M_{25} = I_{11},\ M_{26} = I_{12}, \\ M_{33} = I_8, M_{34} = I_9,\ M_{35} = 0,\ M_{36} = 0, \\ M_{44} = I_{10}, M_{45} = 0,\ M_{46} = 0, \\ M_{55} = I_{13}, M_{65} = I_{14},\ M_{66} = I_{15} \end{gathered} \quad (27)$$

Eq. (28) shows a vector of unknowns.

$$\{\Delta\} = \{u_m \quad w_m \quad \phi_{xm} \quad \psi_{xm} \quad \phi_{zm} \quad \psi_{zm}\}^T \quad (28)$$

Mass and stiffness matrices are symmetric matrices. Natural frequencies are obtained from the non-trivial solution of Eq. (21) i.e. $|[K] - \omega^2[M]| = 0$.

The natural frequency (ω) becomes the fundamental frequency at m=1.

4 NUMERICAL RESULTS AND DISCUSSION

Exact analytical solutions to free vibration problems for simply supported FGM sandwich circular beams are obtained using Navier's technique. To the best of the author's knowledge, literature on free vibration analysis of anti-symmetric sandwich circular beams is not available; therefore, the validity of the present theory is proved by applying it to the frequency analysis of anti-symmetric straight FGM sandwich beams (R = ∞). The beam is made up of ceramic (Alumina) and metal (Aluminum).

The material properties of these constituents are E_c = 380 GPa, ρ_c =3960 kg/m3, E_m = 70 GPa, ρ_m =2702 kg/m3. As compared to the modulus of elasticity, the effects of Poisson's ratio are negligible on the fundamental frequencies of FGM circular beams. Therefore, Poisson's ratio of both ceramic and metal is assumed to be the same (μ= 0.3). These material properties of circular beams are varied through the thickness according to the power law. In the case of sigmoid (S-P-FGM) circular beams, the top surface of the beam is metal-rich whereas the bottom surface is ceramic-rich. However, in the case of three-layered FGM sandwich circular beams, both the top and the bottom surfaces of the beam are metal-rich. The stiffness of the beam becomes lesser as the power-law index increases.

To present the numerical values of the fundamental frequencies, the non-dimensional form given in Eq. (29) is used.

$$\bar{\omega} = \frac{\omega L^2}{h} \sqrt{\frac{\rho_m}{E_m}} \quad (29)$$

The following numerical problems are solved in this study to present the fundamental frequencies of FGM circular beams.

1. Free vibration analysis of three-layered anti-symmetric straight FGM sandwich beams.
2. Free vibration analysis of three-layered anti-symmetric FGM sandwich circular beams.
3. Free vibration analysis of two-layered anti-symmetric sigmoid (S-P-FGM) circular beams.

Problem 1: In this problem, the present theory is applied for the free vibration analysis of anti-symmetric straight FGM sandwich beams. Fundamental frequencies of two types of anti-symmetric FGM sandwich (2-1-1, 2-2-1) straight beams are obtained. The numerical results for straight beams are recovered by setting R = ∞ in the mathematical formulation of the circular beam. The non-dimensional fundamental frequencies are obtained for different power-law index (p) and L/h ratios. The present results are compared with those presented by other researchers [35-38].

Table 1 reveals that the present theory predicts the non-dimensional fundamental frequencies in close agreement with other theories. An increase in the power-law index decreases the non-dimensional fundamental frequencies whereas the increase in the

L/h ratio also increases the non-dimensional fundamental frequencies. An increase in the power-law index reduces the stiffness of the circular beam.

Table 2 also shows that the fundamental frequency is higher for the 2-2-1 lamination scheme as compared to the 2-1-1 scheme due to the higher thickness of the isotropic core.

Problem 2: In this problem, the present theory is extended for the free vibration analysis of anti-symmetric FGM sandwich circular beams. Layerwise thickness coordinates for FGM sandwich circular beams are as follows.

2-1-1: The thickness of layer 1 is double of thicknesses of layer 2 and layer 3 ($h/2$: $h/4$: $h/4$)

2-2-1: The thickness of layer 3 is half of the thicknesses of layer 1 and layer 2 ($2h/5$: $2h/5$: $h/5$)

The fundamental frequencies are obtained for different values of the power-law index and the radius of curvature.

Table 2 summarizes the numerical values of fundamental frequencies for anti-symmetric FGM sandwich circular beams. These are the benchmark results presented by the authors for the first time. The lower value of the radius of curvature shows circular beam has deep curvature whereas the higher value of it represents the circular beam has shallow curvature. The non-dimensional fundamental frequency increases with respect to the increase in the radius of curvature whereas decreases with respect to the increase in the power-law index value. Figs. 4 and 5 show the variations of fundamental frequencies with respect to the radius of curvature for anti-symmetric FGM sandwich (2-1-1 and 2-2-1) circular beams respectively.

Problem 3: In this problem, fundamental frequencies of two-layered anti-symmetric sigmoid (S-P-FGM) circular beams are presented for the first time. Layerwise thickness coordinates for S-P-FGM sandwich circular beams are as follows.

1-0-1: Thicknesses of layers 1 and 3 are the same, and the thickness of layer 2 is zero (h/2: 0: h/2).

The top of the circular beam is metal-rich whereas the bottom of the circular beam is ceramic-rich. The material properties of the sigmoid circular beam are varying anti-symmetrically through the thickness. The numerical values of fundamental frequencies for this circular beam are shown in Table 3.

Examination of Table 3 shows that the values of fundamental frequencies increase with respect to an increase in the value of the radius of curvature. This shows that the shallow circular beam predicts higher values of fundamental frequencies and the deep circular beam predicts lower values of fundamental frequencies. Fig. 6 shows variations of fundamental frequencies with respect to the radius of curvature for sigmoid (S-P-FGM) FGM circular beams.

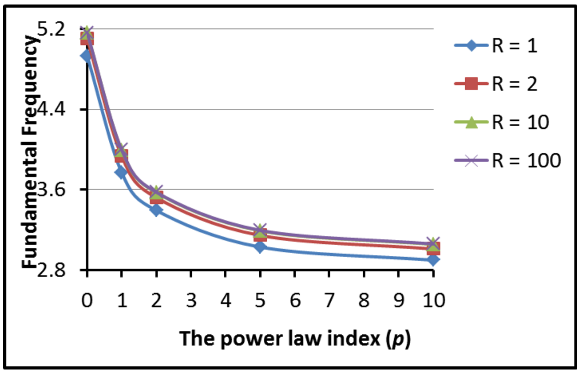

Figure 4. Variations of the fundamental frequency with respect to the power-law indices of functionally graded sandwich (2-1-1) circular beams ($L/h = 5$)

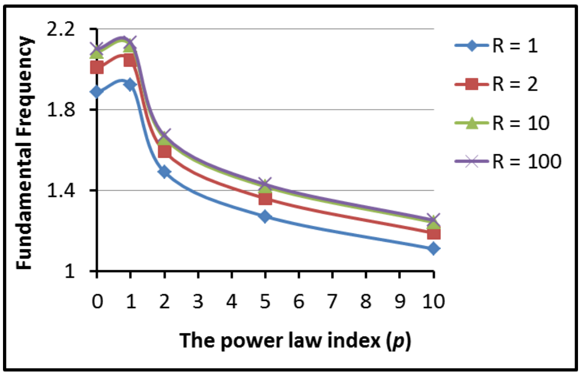

Figure 5. Variations of the fundamental frequency with respect to the power-law indices of functionally graded sandwich (2-2-1) circular beams ($L/h = 5$)

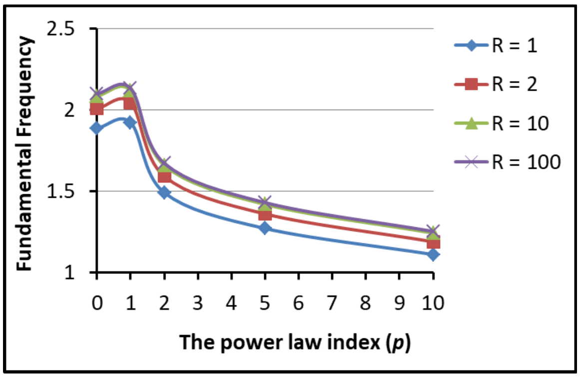

Figure 6. Variations of the fundamental frequency with respect to the power-law indices of sigmoid (S-P-FGM) circular beams ($L/h = 5$)

Table 1 Non-dimensional fundamental frequencies of anti-symmetric straight FGM sandwich beams (Type A, $R = \infty$)

p	**Theory**	***L/h*=5**		***L/h*=20**	
		2-1-1	**2-2-1**	**2-1-1**	**2-2-1**
0	Present	5.1603	5.1603	5.4609	5.4609
	Nguyen et al. [38]	5.1620	5.1620	5.4611	5.4611
	Nguyen et al. [37]	5.1528	5.1528	5.4603	5.4603
	Vo. et al. [36]	5.1618	5.1618	5.4610	5.4610
	Vo et al. [35]	5.1528	5.1528	5.4603	5.4603
1	Present	3.8247	3.9951	3.9737	4.1570
	Nguyen et al. [38]	3.8318	4.0018	3.9824	4.1643
	Nguyen et al. [37]	3.8206	3.9911	3.9775	4.1603
	Vo. et al. [36]	3.8301	4.0005	3.9822	4.1641
	Vo et al. [35]	3.8187	3.9896	3.9774	4.1602
2	Present	3.3551	3.5743	3.4678	3.6989
	Nguyen et al. [38]	3.3685	3.5848	3.4842	3.7051
	Nguyen et al. [37]	3.3546	3.5719	3.4756	3.7049
	Vo. et al. [36]	3.3656	3.5825	3.4838	3.7118
	Vo et al. [35]	3.3514	3.5692	3.4754	3.7049
5	Present	2.9743	3.1949	3.0655	3.2927
	Nguyen et al. [38]	2.9955	3.2122	3.0899	3.3030
	Nguyen et al. [37]	2.9790	3.1966	3.0776	3.3028
	Vo. et al. [36]	2.9912	3.2087	3.0891	3.3133
	Vo et al. [35]	2.9746	3.1928	3.0773	3.3028
10	Present	2.8650	3.0589	2.9535	3.1495
	Nguyen et al. [38]	2.8886	3.0797	2.9797	3.1739
	Nguyen et al. [37]	2.8716	3.0630	2.9665	3.1616
	Vo. et al. [36]	2.8839	3.0588	2.9786	3.1732
	Vo et al. [35]	2.8669	3.0757	2.9662	3.1613

Table 2 Non-dimensional fundamental frequencies of anti-symmetric FGM sandwich circular beams (Type A)

	L/h	*R/h*	*p*				
			0	**1**	**2**	**5**	**10**
2-1-1	5	1	4.9307	3.6357	3.1830	2.8179	2.7140
		2	5.1000	3.7700	3.3036	2.9265	2.8186
		5	5.1505	3.8134	3.3437	2.9633	2.8543
		10	5.1578	3.8208	3.3510	2.9702	2.8610
		20	5.1597	3.8232	3.3534	2.9726	2.8633
		50	5.1602	3.8242	3.3545	2.9737	2.8644
		100	5.1602	3.8245	3.3548	2.9740	2.8648
	20	1	5.2047	3.7817	3.2984	2.9146	2.8079
		2	5.3933	3.9216	3.4213	3.0237	2.9131
		5	5.4499	3.9645	3.4594	3.0578	2.9460
		10	5.4582	3.9711	3.4653	3.0632	2.9512
		20	5.4602	3.9729	3.4670	3.0647	2.9527
		50	5.4608	3.9735	3.4676	3.0653	2.9533
		100	5.4609	3.9736	3.4677	3.0654	2.9534
2-2-1	5	1	4.9307	3.7691	3.3929	3.0278	2.8977
		2	5.1000	3.9322	3.5206	3.1442	3.0096
		5	5.1505	3.9826	3.5627	3.1833	3.0476
		10	5.1578	3.9910	3.5702	3.1906	3.0547
		20	5.1597	3.9936	3.5727	3.1931	3.0572
		50	5.1602	3.9947	3.5738	3.1943	3.0583
		100	5.1602	3.9949	3.5741	3.1946	3.0586
	20	1	5.2047	3.9086	3.5188	3.1310	2.9944
		2	5.3933	4.0930	3.6497	3.2481	3.1065
		5	5.4499	4.1568	3.6901	3.2845	3.1415
		10	5.4582	4.1540	3.6964	3.2903	3.1471
		20	5.4602	4.1561	3.6981	3.2919	3.1487
		50	5.4608	4.1568	3.6987	3.2925	3.1492
		100	5.4609	4.1569	3.6989	3.2926	3.1494

Table 3 Non-dimensional fundamental frequencies of anti-symmetric sigmoid (S-P-FGM) circular beams (Type B)

L/h	*R/h*	*p*				
		0	**1**	**2**	**5**	**10**
5	1	1.8881	1.9214	1.4909	1.2709	1.1104
	2	2.0091	2.0441	1.5922	1.3602	1.1890
	5	2.0690	2.1024	1.6439	1.4056	1.2301
	10	2.0861	2.1186	1.6591	1.4189	1.2424
	20	2.0940	2.1261	1.6662	1.4251	1.2482
	50	2.0986	2.1303	1.6704	1.4287	1.2516
	100	2.1001	2.1317	1.6717	1.4299	1.2527
	∞	2.1016	2.1331	1.6731	1.4311	1.2538
20	1	1.9389	1.9337	1.4994	1.2349	1.0994
	2	2.0273	2.0214	1.5703	1.2945	1.1529
	5	2.0608	2.0542	1.5980	1.3180	1.1742
	10	2.0682	2.0613	1.6043	1.3234	1.1792
	20	2.0711	2.0641	1.6069	1.3256	1.1813
	50	2.0726	2.0655	1.6082	1.3268	1.1824
	100	2.0731	2.0659	1.6086	1.3271	1.1827
	∞	2.0735	2.0663	1.6090	1.3275	1.1830

5 CONCLUSIONS

A fifth-order circular beam theory is developed and applied for the fundamental frequency analysis of anti-symmetric FGM sandwich circular beams. The theory is formulated using Hamilton's principle and Eigenvalue problems are solved using the exact analytical solution technique suggested by Navier. Fundamental frequencies of two-layered anti-symmetric S-P-FGM circular beams and three-layered anti-symmetric FGM sandwich circular beams are obtained for various values of radius of curvatures and the power-law index. Following are the important findings of the present study.

1) The present theory is also displacement-based beam theory like others available in the literature and considers the effects of transverse shear and normal deformations. However, the fifth-order expansion of thickness coordinates is used for the first time in the presented theory.
2) The present theory considers the influence of both transverse shear and normal strains.
3) The frequency analysis of S-P-FGM circular beams is presented in this study for the first time. Therefore, these results can be served as a benchmark for future researchers to compare their studies.

Based on the numerical results and the discussion, it is concluded that the present theory is in excellent agreement with other theories while predicting the fundamental frequencies of straight beams. For the same length and thickness, the value of non-dimensional fundamental frequency increases as the radius of curvature is decreased. Also, the non-dimensional frequency decreases as the power-law index increases.

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