Effect of Slow Parameter Variations on the Vibrations of a Duffing Equation

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Abstract: - Free and forced damped vibrations of a Duffing equation with cubic nonlinearities is considered. The damping, nonlinearity and external excitation parameters are assumed to vary slowly in time. Using the Method of Multiple Scales, a perturbation technique, the amplitude and phase modulation equations are derived in its most general case. First, the free vibration case is treated. Decaying, built-up and harmonically varying functions are taken to model the slow variations of the parameters in time. The amplitude and phase modulation equations can be integrated to obtain closed form solutions in general for free vibrations. For the forced vibrations, a resort to the numerical techniques is required for the integration of the amplitude and phase modulation equations. It is shown that slow variations on the coefficients may lead to substantial changes in the dynamics of the problem.

Keywords: - Nonlinear Vibrations, Free and Forced Vibrations, Method of Multiple Scales, Transient Solutions, Slow Parameter Variation.

1. INTRODUCTION

In real world problems, oscillatory behavior is commonly encountered whether the system is mechanical, electrical or quantum. Continuous systems (beam, cable, string), electrical circuits, alternating currents exhibit oscillatory behavior when excited by an external agent. The Duffing equation is one of the most fundamental mathematical models in representing nonlinear oscillatory behavior. A vibrating mass attached to a cubic nonlinear spring subject to external harmonic excitation is the simplest derivation of the Duffing equation. Discrete systems as well as continuous systems when reduced to ordinary differential equation can be modelled by the Duffing equation. For the former, the famous example is the motion of a pendulum and for the latter, one of the well-known examples is the nonlinear beam vibrations after discretization of the partial differential equation. In its general form, the Duffing equation inherits an inertia term, a damping, a linear and a nonlinear restoring force term and an external excitation term. Many variants of the Duffing equation exists by specifying the nonlinear restoring force of the system.

Free and forced vibrations of the nonlinear equation was extensively studied for cubic [1-14], cubic-quintic [13, 15-20], quintic [13, 21], quadratic and cubic [11, 13, 14, 22-24], cubic-quintic-septic [25-27], a more general restoring force [28, 29]. Various methods were employed in search of analytical solutions such as the Method of Multiple Scales [1, 11, 18, 21, 22, 24, 27, 30-33], the Multiple Scales Lindstedt Poincare Method [1, 7, 8, 19, 21, 22], Harmonic Balance Method [15, 23, 30], Lindstedt Poincare Method [28, 30], Parameter Splitting Multiple Scales Method [2, 29], Modified Differential Transform Method [3], Homotopy Analysis Method [4], Jacobi Elliptic Functions [5, 6, 16], Fourier Coefficients [17], Method of Surplus and Deficiency [25], Laplace Decomposition Algorithm [9], Taylor Wavelets [10], Optimal Homotopy Asymptotic Method [12], Perturbation Iteration Method [13], and Shift-Perturbation Method [14].

In a very recent study, numerical solutions of the Duffing equation were feed in a neural network algorithm for training in order to reduce the computational costs [34]. Chaotic motion of the Duffing oscillator was also investigated [35].

All the mentioned studies dealt with the models possessing constant physical parameters. In this work, for the first time, the slow variations in time of the physical parameters are considered. The mathematical model consists of an inertia term, damping term, linear and cubic nonlinear restoring force term and an external excitation term. The physical problem is considered to be in dimensionless form more suitable for a perturbational analysis [30-33]. Damping coefficient, cubic nonlinearity coefficient and the external excitation coefficient are assumed to vary slowly in time. Variation functions of decaying type, built-up type and harmonic fluctuation type are considered. To the best of the author's knowledge, such slow variations in the physical parameters of the Duffing system were not examined before. Since transient solutions are quite important in the problem investigated, methods such as renormalization method, Lindstedt Poincare method are not suitable since they produce only the steady state solutions. [30-32]. Hence, the Method of Multiple Scales, known to effectively determine transient solutions, is employed in search of analytical solutions [30-32]. The equations governing the modulation of amplitude and phases are derived in its most general case. Free and forced vibration cases are investigated separately. For the case of free vibrations, closed form functional expressions are available for most of the time for the amplitudes and phases by directly integrating the amplitude-phase modulation equations. In contrast, for the case of forced vibrations, closed form transient solutions are unavailable which requires numerical integration of the amplitude-phase equations. The main concern in this study is to retrieve the transient solutions, not the steady state solutions. For the forced vibrations with built up coefficients, in the long run, the steady state solutions given in the literature [30] is retrieved. For the forced vibrations with harmonically fluctuating parameters, the existence of quasi-steady-state solutions are depicted with the aid of figures. Some complex phenomena which were not reported before are retrieved for the fluctuating coefficients having sign changes. It is shown that slow variations may affect substantially the vibrational behavior qualitatively and quantitatively.

2. AMPLITUDE AND PHASE MODULATIONS

Consider the dimensionless damped forced Duffing equation with a cubic nonlinearity

$$\ddot{u} + 2\varepsilon\mu\dot{u} + u + \varepsilon k u^3 = \varepsilon f cos\Omega t \quad , \qquad (1)$$

where the damping (μ) , nonlinearity (k) and excitation amplitude (f) parameters are assumed to be slowly varying in time, i.e.,

$$\mu = \mu(\varepsilon t), \quad k = k(\varepsilon t), \quad f = f(\varepsilon t) \quad , \qquad (2)$$

with $\varepsilon \ll 1$ is a small perturbation parameter. Instead of using multiple perturbation parameters, it is advantageous to use a single perturbation parameter [33] and reorder the damping, nonlinearity and excitation in terms of this single perturbation parameter. One of the derivations of the above equation for constant parameters may be found for a Holmes-Duffing oscillator [35]. The primary resonance case is considered in which the external excitation frequency Ω is near to the natural frequency of the system

$$\Omega = 1 + \varepsilon\sigma \quad , \qquad (3)$$

where σ is the detuning parameter expressing the nearness of the external frequency to the natural frequency. The damping, nonlinear and external excitation terms are reordered so that their effects counterbalance each other at the last level of approximation which is the usual ordering for primary resonances.

An approximate solution of (1)-(3) will be searched by employing the Method of Multiple Scales, a perturbation technique which successfully determines the transient solutions [30-32]. The method utilizes the principle that in dynamical systems, some changes occur fast and others slower. Analyzing the problem in different time scales enables one to eliminate unphysical blow up type singular solutions. A two-term expansion is written

$$u(t;\varepsilon) = u_0(T_0,T_1) + \varepsilon u_1(T_0,T_1) + O(\varepsilon^2) \quad , \qquad (4)$$

where $T_0 = t$ is the fast time scale and $T_1 = \varepsilon t$ is the slow time scale. The time derivatives are

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \cdots, \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots . \qquad (5)$$

Inserting (4) and (5) into (1) in view of (2) and separating each order yields

$$D_0^2 u_0 + u_0 = 0 \qquad (6)$$

$$D_0^2 u_1 + u_1 = -2D_0 D_1 u_0 - 2\mu(T_1) D_0 u_0 - k(T_1) u_0^3 + f(T_1) cos\Omega T_0 \quad . \qquad (7)$$

The solution at the first order is

$$u_0 = A(T_1)e^{iT_0} + cc = a(T_1)\cos\left(t + \beta(T_1)\right) , \qquad (8)$$

where

$$A(T_1) = \frac{1}{2} a(T_1) e^{i\beta(T_1)} \quad , \qquad (9)$$

and *cc* stands for the complex conjugates of the preceding terms. Inserting (8) and (3) into the next level of approximation and rearranging

$$D_0^2 u_1 + u_1 = e^{iT_0}\left(-2iD_1 A - 2\mu i A - 3kA^2\bar{A} + \frac{f}{2} e^{i\sigma T_1}\right) + cc + NST \quad , \qquad (10)$$

where *NST* stands for non-secular terms. Eliminating the secularities

$$2iD_1 A + 2\mu i A + 3kA^2\bar{A} - \frac{f}{2} e^{i\sigma T_1} = 0 \; . \qquad (11)$$

For free vibrations $f = 0$ and upon substitution of the complex amplitudes from (9) and separating real and imaginary parts, one has for amplitude and phases

$$D_1 a = -\mu(T_1) a \qquad (12)$$

$$D_1 \beta = \frac{3}{8} k(T_1) a^2 \quad . \qquad (13)$$

The approximate solution is then given in (8) with the real amplitudes and phases governed by (12) and (13).

For the forced vibrations ($f \neq 0$), defining a new phase

$$\gamma = \sigma T_1 - \beta \quad , \qquad (14)$$

from the complex amplitude equation (11), one finally obtains

$$D_1 a = -\mu(T_1) a + \frac{f(T_1)}{2} sin\gamma \qquad (15)$$

$$D_1\gamma = \sigma - \frac{3}{8}k(T_1)a^2 + \frac{f(T_1)}{2a}cos\gamma \quad , \qquad (16)$$

which are the amplitude and phase modulation equations for a general duffing equation with slowly varying parameters. Substituting $\beta = \sigma T_1 - \gamma$ into (8) and using (3), the approximate solution for the forced vibrations is

$$u = a(T_1)\cos\big(\Omega t - \gamma(T_1)\big) + O(\varepsilon) \, , \qquad (17)$$

where the amplitudes and phases are governed by (15) and (16).

Free and forced vibrations will be treated separately in the subsequent sections.

3. FREE VIBRATIONS

For free vibrations $f = 0$, and the equations to be solved are (12) and (13) for the amplitudes and phases. Three different variations are considered: Decaying, built-up and harmonic variation.

3.1. Constant Damping, Decaying Nonlinearity

Take

$$\mu = \mu_0 \, , \qquad k = k_0 e^{-\alpha T_1} \quad , \qquad (18)$$

where α is the decaying rate of the nonlinearity. From (12) and (13), the amplitudes and phases are

$$a = a_0 e^{-\varepsilon \mu_0 t} \qquad (19)$$

$$\beta = -\frac{3}{8}\frac{k_0 a_0^2}{\alpha + 2\mu_0} e^{-\varepsilon(\alpha+2\mu_0)t} + \beta_0 \quad , \qquad (20)$$

and the approximate solution in terms of the fast time scale is

$$u = a_0 e^{-\varepsilon\mu_0 t} cos\left(t - \frac{3}{8}\frac{k_0 a_0^2}{\alpha + 2\mu_0} e^{-(\alpha+2\mu_0)\varepsilon t} + \beta_0\right). \qquad (21)$$

As $t \to \infty$ the solution will tend to zero since the amplitude tends to zero and the phase tends to the constant value β_0. A sample time history for the problem is given in Figure 1.

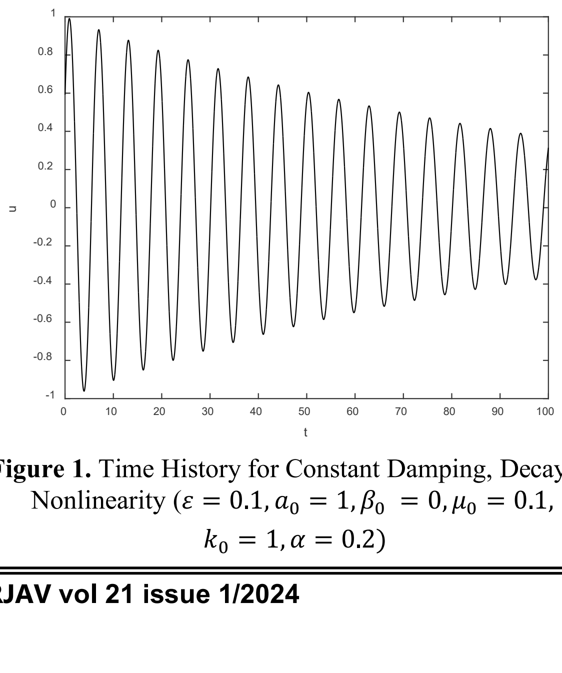

Figure 1. Time History for Constant Damping, Decaying Nonlinearity ($\varepsilon = 0.1, a_0 = 1, \beta_0 = 0, \mu_0 = 0.1, k_0 = 1, \alpha = 0.2$)

3.2. Decaying Damping and Nonlinearity

For this case, both parameters decay in time. If one assumes a similar decay rate for both of them, then

$$\mu = \mu_0 e^{-\alpha T_1} \, , \quad k = k_0 e^{-\alpha T_1} \quad , \qquad (22)$$

where α is the decaying rate of the nonlinearity. The amplitudes and phases are solved from (12) and (13)

$$a = a_0 e^{\frac{\mu_0}{\alpha} e^{-\varepsilon\alpha t}} \qquad (23)$$

$$\beta = -\frac{3}{16}\frac{k_0 a_0^2}{\mu_0} e^{2\frac{\mu_0}{\alpha} e^{-\varepsilon\alpha t}} + \beta_0 \quad , \qquad (24)$$

and the approximate solution is

$$u = a_0 e^{\frac{\mu_0}{\alpha} e^{-\varepsilon\alpha t}} cos\left(t - \frac{3}{16}\frac{k_0 a_0^2}{\mu_0} e^{2\frac{\mu_0}{\alpha} e^{-\varepsilon\alpha t}} + \beta_0\right) \qquad (25)$$

As $t \to \infty$, the amplitude tends to the constant value a_0 and the phase tends to the constant value $\beta_0 - \frac{3}{16}\frac{k_0 a_0^2}{\mu_0}$. If the nonlinearity coefficient is constant, i.e., the decay rate α=0, then the solution corresponding to the usual damped nonlinear system is retrieved [33]. Figure 2 is a numerical plot of the solution.

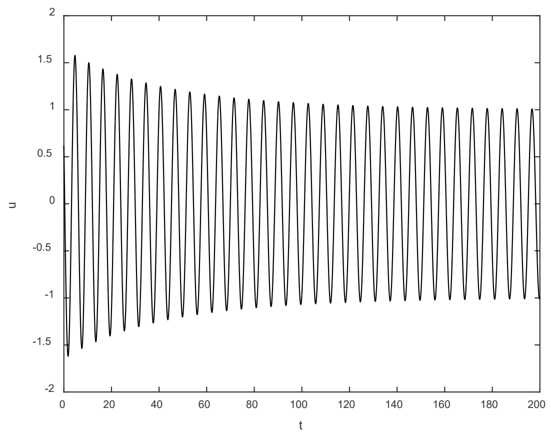

Figure 2. Time History for Decaying Damping and Nonlinearity ($\varepsilon = 0.1, a_0 = 1, \beta_0 = 0, \mu_0 = 0.1, k_0 = 1, \alpha = 0.2$)

As can be seen from the Figure, the amplitude tends to a constant value as time proceeds.

3.3. Constant Damping and Built-Up Nonlinearity

For this case,

$$\mu = \mu_0 \, , \qquad k = k_0(1 - e^{-\alpha T_1}) \quad , \qquad (26)$$

where the nonlinearity gradually builds up in time from 0 to k_0. The amplitudes and phases are solved from (12) and (13)

$$a = a_0 e^{-\mu_0 T_1} \qquad (27)$$

$$\beta = \frac{3}{8}k_0 a_0^2 \left(\frac{1}{2\mu_0 + \alpha} e^{-\varepsilon\alpha t} - \frac{1}{2\mu_0}\right) e^{-2\varepsilon\mu_0 t} + \beta_0 \, , \qquad (28)$$

and the approximate solution is

$$u = a_0 e^{-\mu_0 \varepsilon t} cos\left(t + \frac{3}{8} k_0 a_0^2 \left(\frac{1}{2\mu_0+\alpha} e^{-\varepsilon\alpha t} - \frac{1}{2\mu_0}\right) e^{-2\varepsilon\mu_0 t} + \beta_0\right) . \quad (29)$$

As $t \to \infty$ the amplitude tends to zero and the phase tends to the constant value β_0. If the decaying ratio is large, i.e. $\alpha \to \infty$, the usual nonlinear damped Duffing solution is obtained up to first order approximation [1]. Figure 3 depicts the time history for the problem.

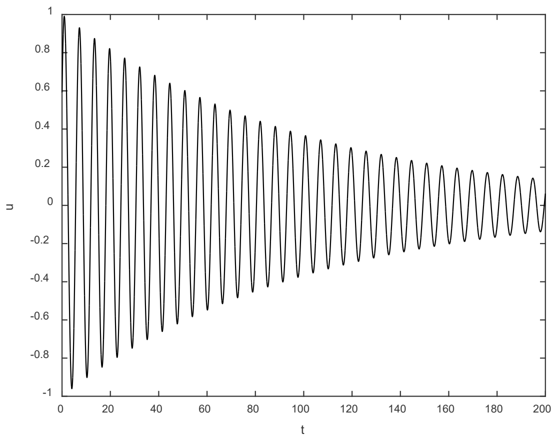

Figure 3. Time History for Constant Damping and Built-Up Nonlinearity ($\varepsilon = 0.1, a_0 = 1, \beta_0 = 0, \mu_0 = 0.1, k_0 = 1, \alpha = 0.2$)

If there is no damping, the built-up nonlinearity leads to

$$u = a_0 cos\left((1 + \varepsilon \frac{3}{8} k_0 a_0^2)t + \frac{3}{8} \frac{k_0 a_0^2}{\alpha} e^{-\varepsilon\alpha t} + \beta_0\right) , \quad (30)$$

for the transient solutions and for a fairly long time, the solution reduces to

$$u = a_0 cos\left((1 + \varepsilon \frac{3}{8} k_0 a_0^2)t + \beta_0\right), \quad (31)$$

which is the well-known approximate solution of the free-undamped Duffing equation [30].

3.4. Oscillating Nonlinearity and No Damping

For this case,

$$\mu = 0 , k = k_0 + k_1 \cos(\varepsilon\lambda t) , \quad (32)$$

where the nonlinearity harmonically varies about a mean value. The amplitudes and phases are solved from (12) and (13)

$$a = a_0 \quad (33)$$

$$\beta = \varepsilon \frac{3}{8} k_0 a_0^2 t + \frac{3}{8} \frac{a_0^2 k_1}{\lambda} \sin(\varepsilon\lambda t) + \beta_0 , \quad (34)$$

and the approximate solution is

$$u = a_0 cos\left(\left(1 + \varepsilon \frac{3}{8} k_0 a_0^2\right) t + \frac{3}{8} \frac{a_0^2 k_1}{\lambda} \sin(\varepsilon\lambda t) + \beta_0\right) . \quad (35)$$

The nonlinear frequency for the problem is $\left(1 + \varepsilon \frac{3}{8} k_0 a_0^2\right)$ and the phases fluctuate in time without reaching a constant value. When the harmonic component of nonlinearity is absent, the usual undamped Duffing solution given in equation (31) is again retrieved [30]. Figure 4 is a plot of the time history for the problem.

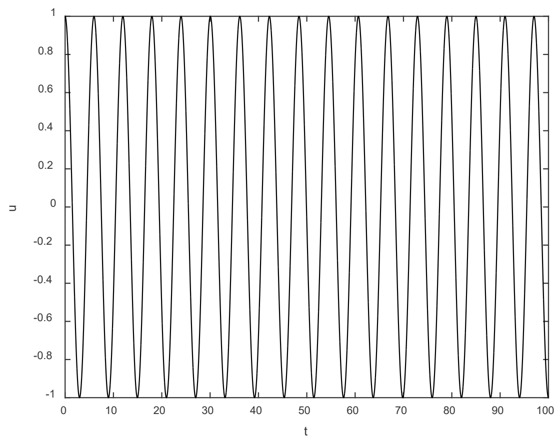

Figure 4. Time History for No Damping and Oscillating Nonlinearity ($\varepsilon = 0.1, a_0 = 1, \beta_0 = 0, k_0 = 1$ $k_1 = 0.4, \lambda = 0.6$)

3.5. Constant Damping and Oscillating Nonlinearity

For this case,

$$\mu = \mu_0 , \qquad k = k_0 + k_1 \cos(\varepsilon\lambda t), \quad (36)$$

The amplitudes and phases are

$$a = a_0 e^{-\varepsilon\mu_0 t} \quad (37)$$

$$\beta = e^{-2\varepsilon\mu_0 t}\left(-\frac{3}{16}\frac{k_0 a_0^2}{\mu_0} + \frac{3}{8}\frac{k_1 a_0^2}{\lambda^2+4\mu_0^2}(\lambda \sin(\varepsilon\lambda t) - 2\mu_0 cos(\varepsilon\lambda t)\right) + \beta_0 , \quad (38)$$

and the approximate solution is

$$u = a_0 e^{-\varepsilon\mu_0 t} cos\left(t + e^{-2\varepsilon\mu_0 t}\left(-\frac{3}{16}\frac{k_0 a_0^2}{\mu_0} + \frac{3}{8}\frac{k_1 a_0^2}{\lambda^2+4\mu_0^2}(\lambda \sin(\varepsilon\lambda t) - 2\mu_0 cos(\varepsilon\lambda t)\right) + \beta_0\right) . \quad (39)$$

The amplitude tends to zero and the phase tends to a constant value β_0 for long time intervals. When the harmonic oscillations of the nonlinearity are not present, i.e. $k_1 = 0$, the usual damped-Duffing solution without external excitation is retrieved up to first order approximation [1]. Figure 5 is a plot of the time history for the problem.

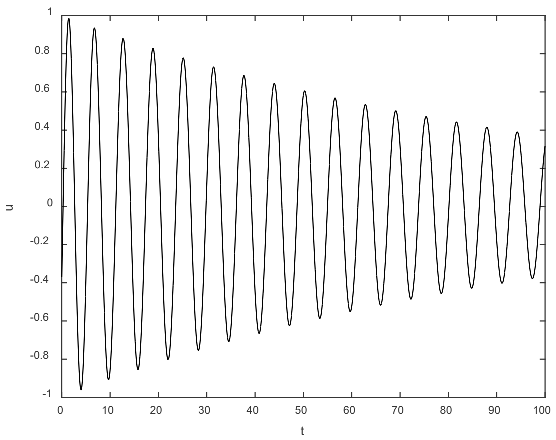

Figure 5. Time History for Constant Damping and Oscillating Nonlinearity ($\varepsilon = 0.1, a_0 = 1, \beta_0 = 0, k_0 = 1$ $k_1 = 0.4, \lambda = 0.6, \mu_0 = 0.1$)

4. FORCED VIBRATIONS

For forced-vibrations $f \neq 0$, and the equations to be solved are (15) and (16) for the amplitudes and phases. Two cases are considered: Built-up and harmonic variation.

4.1. Built-Up Damping, Nonlinearity and Excitation

For this case,

$$\mu = \mu_0(1 - e^{-\varepsilon\alpha_1 t}), \quad k = k_0(1 - e^{-\varepsilon\alpha_2 t})$$
$$f = f_0(1 - e^{-\varepsilon\alpha_3 t}) \quad . \tag{40}$$

Substituting into (15) and (16), in the long run, the coefficients will tend to their limiting values of μ_0, k_0 and f_0. The steady state solutions would then be the well-known frequency response solution for the Duffing equation [30, 31,7]

$$\Omega = 1 + \varepsilon\left(\frac{3}{8}k_0 a_0^2 \mp \sqrt{\frac{f_0^2}{4a_0^2} - \mu_0^2}\right). \tag{41}$$

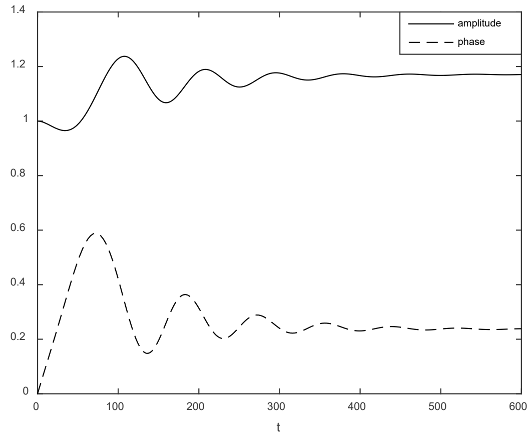

Figure 6. Time History of Amplitude and Phases for Built-Up Parameters ($\varepsilon = 0.1, a_0 = 1, \beta_0 = 0, \mu_0 = 0.1, f_0 = 1, k_0 = 1, \alpha_1 = 0.2, \alpha_2 = 0.15, \alpha_3 = 0.1$)

Since we are concerned mainly with the transient solutions, equations (15) and (16) are numerically integrated and results are shown in Figure 6.

The amplitude and phases have oscillating transient solutions which tend to constant values after sufficient time passes. Figure 7 is a direct plot of the amplitudes versus the phases

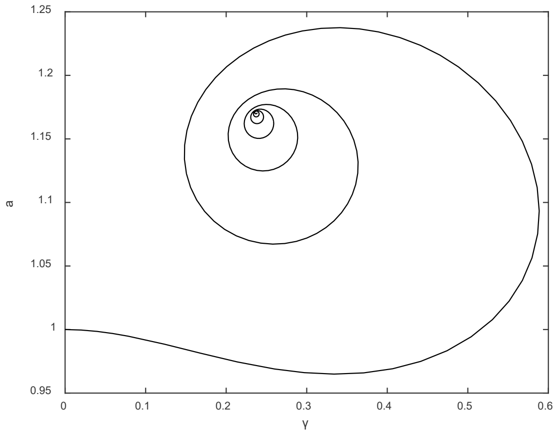

Figure 7. Amplitude versus Phases for Built-Up Parameters ($\varepsilon = 0.1, a_0 = 1, \beta_0 = 0, \mu_0 = 0.1, f_0 = 1, k_0 = 1, \alpha_1 = 0.2, \alpha_2 = 0.15, \alpha_3 = 0.1$)

Starting from the initial point (0,1) the amplitude and phase converges to the steady state point in the figure.

4.2. Harmonically Varying Damping, Nonlinearity and Excitation

Assume now that the physical parameters are harmonically varying about a constant mean value

$$\mu = \mu_0 + \mu_1 \cos(\varepsilon\lambda_1 t), \quad k = k_0 + k_1 \cos(\varepsilon\lambda_2 t)$$
$$f = f_0 + f_1 \cos(\varepsilon\lambda_3 t) \quad . \tag{42}$$

Numerical integrations of (15) and (16) are performed for employing (42) for the parameters. Figure 8 depicts the change of amplitudes and phases in time.

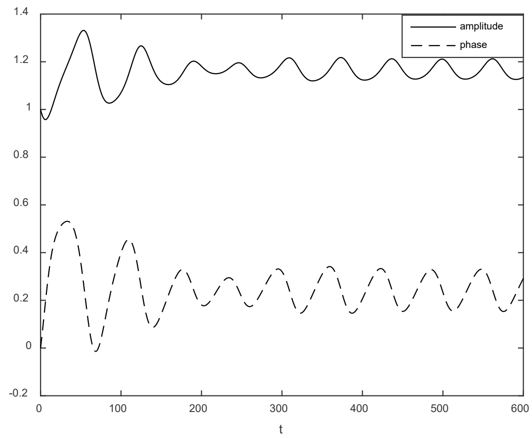

Figure 8. Time History of Amplitude and Phases for Harmonically Varying Parameters ($\varepsilon = 0.1,\ a_0 = 1,\ \beta_0 = 0,\ \mu_0 = 0.1,\ \mu_1 = 0.04,\ k_0 = 1,\ k_1 = 0.4,\ f_0 = 1, f_1 = 0.4, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \sigma = 0.1$)

The amplitude and phases do not tend to a constant value, rather they oscillate about a mean value in the long run. We can name this behavior as a quasi-steady-state behavior. Figure 9 is a direct plot of the amplitudes versus the phases.

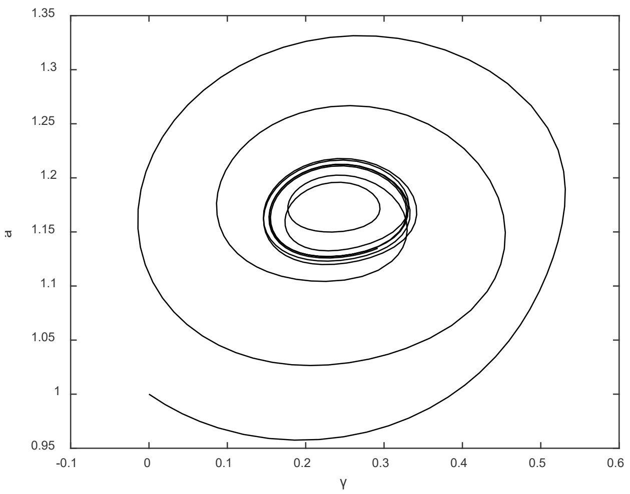

Figure 9. Amplitude versus Phases for Harmonically Varying Parameters ($\varepsilon = 0.1,\ a_0 = 1,$ $\beta_0 = 0,\ \mu_0 = 0.1,\ \mu_1 = 0.04,\ k_0 = 1,\ k_1 = 0.4,$ $f_0 = 1, f_1 = 0.4, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \sigma = 0.1$)

The quasi-steady state behavior is well reflected in the figure.

A more complex behavior can also be observed when the nonlinearity slowly changes sign in time ($k_0 = 0$). One such behavior is depicted in Figure 10.

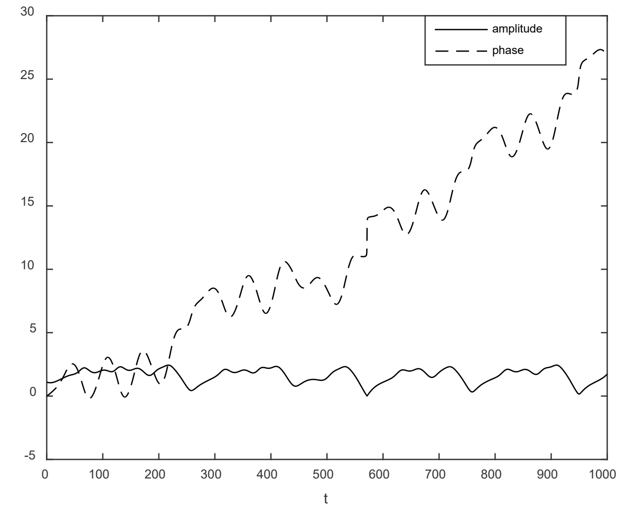

Figure 10. Time History of Amplitude and Phases for Harmonically Varying Parameters with a Sign Change in Non-linearity ($\varepsilon = 0.1,\ a_0 = 1.1, \beta_0 = 0,$ $\mu_0 = 0.1,\ \mu_1 = 0.04,\ k_0 = 0,\ k_1 = 1, f_0 = 1,$ $f_1 = 0.4, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \sigma = 0.1$)

The amplitudes and phases display complex behavior. While the amplitudes stay in a limited region, the phases grow in an oscillating manner. The direct relationship for the amplitudes and phases cannot be classified as simple motions (Figure 11).

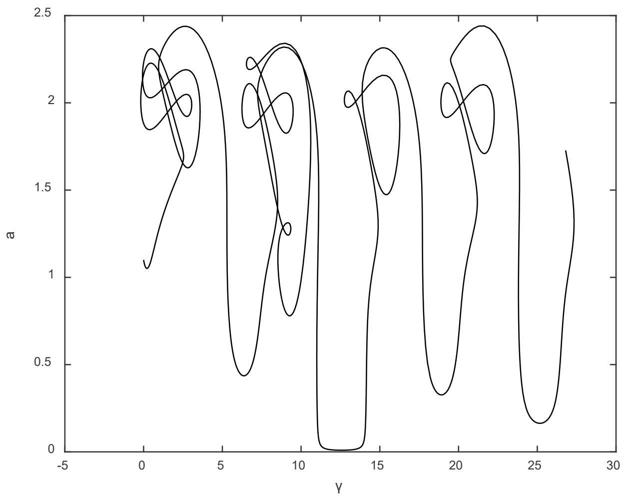

Figure 11. Amplitude versus Phases for Harmonically Varying Parameters with a Sign Change in Non-linearity ($\varepsilon = 0.1,\ a_0 = 1.1, \beta_0 = 0,\ \mu_0 = 0.1,\ \mu_1 = 0.04,$ $k_0 = 0,\ k_1 = 1, f_0 = 1, f_1 = 0.4, \lambda_1 = 1, \lambda_2 = 1,$ $\lambda_3 = 1, \sigma = 0.1$)

If we change the initial condition to $a_0 = 1.2$ with all other parameters remaining the same, the system behavior turns out to be one of a quasi-steady-state. Generally speaking, the variations in the physical parameters depicted in Figures 6-11 result in oscillation type behavior for the amplitudes and phases leading to steady state or quasi-steady state solutions. If the nonlinear parameter changes sign, then the responses become much more complex.

5. CONCLUSIONS

Real vibrational systems are non-ideal in the sense that their physical parameters may not remain constant during oscillations. The case of slow variations in the physical parameters of the Duffing equation which arises in many nonlinear dynamical problems is considered in this work. Transient solutions of an externally excited, damped equation with cubic nonlinearity is considered. The system parameters, namely the damping, nonlinearity and excitation amplitude, are assumed to change slowly in time. Three different variation functions are used in this study: Decaying, Built-Up, Harmonic functions.

Approximate analytical solutions are found using the Method of Multiple Scales which is known to retrieve successfully the transient solutions. Analytical solutions of the amplitude-phase modulation equations can be found by direct integration for most of the cases of free vibrations. For externally excited systems, closed form solutions of the amplitude-phase modulation equations do not exist, hence, the equations are solved numerically. Steady-state behavior, quasi-steady-state behavior as well as more complex behavior is observed for such systems. Results are compared with the existing results in the literature for the degenerate cases of constant parameters.

It is found that slow variations in the parameters of nonlinear systems may have substantial changes in the dynamics of the problem. Depending on the assumed type of variation, the amplitudes and phases may exhibit, decaying, built up and oscillatory behavior. Steady-state periodic, quasi steady-state periodic and more complex behavior can be obtained from the parameter variations.

Future work may include slow variations in physical parameters of other nonlinear dynamical problem models. This work may be used as a fundamental reference for further relevant studies and may outline the necessary algorithms for handling such problems.

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