Three Points Frictionless Simultaneous Collision of Rigid Solid

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Abstract: - This paper deals with the simultaneous frictionless three-point collision of a rigid solid. The working hypotheses and the conditions under which the problem can be solved are presented. The impulses at the contact points, the velocity of the rigid body after the collision and its energy variation are determined. Some special cases are also discussed. The theory is illustrated based on a few examples. The paper concludes with conclusions and future directions for study.

Keywords: - Collision, simultaneous, multi-point, coefficient of restitution.

1. INTRODUCTION

The actual study of the problem is presented in our previous paper [26], based on the references [1 – 25]. Some aspects must be remembered here [26]:

– simultaneous vanishing of the normal velocities in the contact points;

– there is no jamb phenomenon.

In these conditions (which are necessary, but not sufficient ones) the reader may obtain the velocities after collision and the impulses.

The study of the simultaneous collisions of the rigid solid at two points highlighted the following aspects:

– one can does not study the simultaneous collisions as a successive one, that is, firstly the collision takes place at a certain point and, after this collision ends, the new collision will take place at another point;

– it is not valid the formula which links the normal velocity of the contact point after and before the collision, that is $k=-\frac{v_{12n}}{v_{12n}^{0}}$, where the superior index 0 marks the situation before the collision;

– in the collision with friction one must use the energetic coefficient of restitution. In the conditions of the collision without friction it is not important which coefficient of restitution is used, the results being identical;

– the simultaneous collision are described only for particular cases of rigid solids or kinematic chains;

– in general, coefficients of restitution are considered to have the same value at all collision points.

The simultaneous collisions lead to the satisfaction of Euler's distribution of velocities for a rigid solid.

In the previous paper we have discussed some aspects concerning the simultaneous collisions. In this paper we consider two rigid bodies with bilateral constraints which simultaneously collide at three points.

2. SIMULTANEOUS COLLISION OF TWO RIGID SOLIDS WITH CONSTRAINTS

We will use the same notations as in [26]. These notations are:

– $\{\mathbf{v}^{(1)}\}$, $\{\mathbf{v}^{(2)}\}$ – the matrices of velocities of the two rigid solids after the collision;

– $\{\mathbf{v}^{(0)1}\}$, $\{\mathbf{v}^{0(2)}\}$ – the matrices of velocities of the two rigid solids before the collision,

– C_1, C_2 – the centers of weight of the two rigid solids,

– O_1, O_2 – the points at which the two rigid solids have bilateral constraints;

– m_1, m_2 – the masses of the two rigid solids;

– J_{x_1}, J_{y_1}, J_{z_1}, J_{x_2}, J_{y_2}, J_{z_2} – the moments of inertia of the two rigid solids;

– $[\boldsymbol{\eta}]$ – the matrix given by

$$[\boldsymbol{\eta}]=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{bmatrix}; \quad (1)$$

– A_1, A_2, A_3 – the points at which simultaneous collision takes place;

– $[\mathbf{S}_1]$, $[\mathbf{S}_2]$ – the matrices of the screw coordinates of the simple restriction at the points of bilateral constraints;

– $[\mathbf{Q}_1]$, $[\mathbf{Q}_2]$ – the matrices of the screw coordinates of the possible motions of the two rigid solids;

– $[\mathbf{U}_1]$, $[\mathbf{U}_2]$ – the matrices of screw coordinates of the support lines of the impulses;

– $\{\mathbf{P}\}$ – the column matrix of the impulses.

In a similar way to the simultaneous collisions at two points of a rigid solid with bilateral contraints, one may write

$$\{\mathbf{v}^{(1)}\}-\{\mathbf{v}^{0(1)}\} = [\mathbf{Q}_1][\mathbf{M}_{1red}]^{-1}[\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_1]\{\mathbf{P}\}, \quad (2)$$

for the first rigid solid, and

$$\{\mathbf{v}^{(2)}\}-\{\mathbf{v}^{0(2)}\} = [\mathbf{Q}_2][\mathbf{M}_{2red}]^{-1}[\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_2]\{\mathbf{P}\}, \quad (3)$$

for the second one.

The previous expressions are multiplied at the left side by $[\mathbf{U}_1]^{\mathrm{T}}[\boldsymbol{\eta}]$ and $[\mathbf{U}_2]^{\mathrm{T}}[\boldsymbol{\eta}]$ obtaining

$$\{\mathbf{v}_n^{(1)}\}-\{\mathbf{v}_n^{0(1)}\} = [\mathbf{U}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_1][\mathbf{M}_{1red}]^{-1}[\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_1]\{\mathbf{P}\}, \quad (4)$$

$$\{\mathbf{v}_n^{(2)}\}-\{\mathbf{v}_n^{0(2)}\} = [\mathbf{U}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_2][\mathbf{M}_{2red}]^{-1}[\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_2]\{\mathbf{P}\}. \quad (5)$$

It results

$$[\mathbf{U}_i]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_i][\mathbf{M}_{ired}]^{-1}[\mathbf{Q}_i]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_i]=[\mathbf{G}_i], \; i=1,2, \quad (6)$$

where

$$\{\mathbf{v}_{12n}\}-\{\mathbf{v}_{12n}^0\}=-[[\mathbf{G}_1]+[\mathbf{G}_2]]\{\mathbf{P}\}. \quad (7)$$

In addition, one has

$$\{\mathbf{v}_{12n}\}-\{\mathbf{v}_{12n}^0\}=-[[\mathbf{I}]+[\mathbf{K}]]\{\mathbf{v}_{12n}^0\}, \quad (8)$$

with

$$[\mathbf{I}]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad (9)$$

and

$$[\mathbf{K}]=\begin{bmatrix}k_1&0&0\\0&k_2&0\\0&0&k_3\end{bmatrix}. \quad (10)$$

We also get

$$\{\mathbf{P}\}=[[\mathbf{G}_1]+[\mathbf{G}_2]]^{-1}[[\mathbf{I}]+[\mathbf{K}]]\{\mathbf{v}_{12n}^0\}. \quad (11)$$

where

$$[\mathbf{M}_{ired}]=[\mathbf{Q}_i]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_i][\mathbf{Q}_i], \; i=1,2, \quad (12)$$

$$\{\mathbf{v}_n^{0(i)}\}=[\mathbf{U}_i]^{\mathrm{T}}[\boldsymbol{\eta}]\{\mathbf{v}^{0(i)}\}, \; i=1,2, \quad (13)$$

$$\{\mathbf{v}_{12n}^0\}=\{\mathbf{v}_n^{0(1)}\}-\{\mathbf{v}_n^{0(2)}\}. \quad (14)$$

The velocities after the collision are

$$\{\mathbf{v}^{(1)}\}=\{\mathbf{v}^{0(1)}\} -[\mathbf{Q}_1][\mathbf{M}_{1red}]^{-1}[\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_1]\{\mathbf{P}\} \quad (15)$$

and

$$\{\mathbf{v}^{(2)}\}=\{\mathbf{v}^{0(2)}\} +[\mathbf{Q}_2][\mathbf{M}_{2red}]^{-1}[\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_2]\{\mathbf{P}\}. \quad (16)$$

The impulses of constraints are

$$\{\xi_1\}=[[\mathbf{S}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_1]^{-1}[\mathbf{S}_1]]^{-1} [\mathbf{S}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_1]^{-1}[\mathbf{U}_1]\{\mathbf{P}\} \quad (17)$$

and

$$\{\xi_2\}=-[[\mathbf{S}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_2]^{-1}[\mathbf{S}_2]]^{-1} [\mathbf{S}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_2]^{-1}[\mathbf{U}_2]\{\mathbf{P}\}. \quad (18)$$

The matrices of the simple impulses are $[\mathbf{S}_i]\{\xi_i\}$, with $i=1,2$.

3. EXAMPLE

For the frames in Fig. 1, cylindrically jointed at the points O_1 and O_2, having the centers of weight in C_1 and C_2 and colliding at the points A_1, A_2 and A_3, one knows the dimensions a_1, a_2, a_3, a_4 and a_5, the masses m_1 and m_2, the inertial moments J_{x_1}, J_{y_1},

J_{z_1}, and J_{x_2}, J_{y_2}, J_{z_2}, respecively, relative to each frame with respect to the central inertial systems, the coefficients of restitution k_1 at the point A_1. k_2 at the point A_2, and k_3 at the point A_3, and the initial distributions of velocities (the magnitudes of the angular veocities ω_{10} and ω_{20}). The impulses $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$ are situated on the directions $\alpha_1\mathbf{i}+\beta_1\mathbf{j}+\gamma_1\mathbf{k}$, $\alpha_2\mathbf{i}+\beta_2\mathbf{j}+\gamma_2\mathbf{k}$, and $\alpha_3\mathbf{i}+\beta_3\mathbf{j}+\gamma_3\mathbf{k}$, respectively.

One asks for the impulses at the collision points A_1, A_2, and A_3, the constraint impulses at the points O_1 and O_2, and the distributions of velocities after the collision.

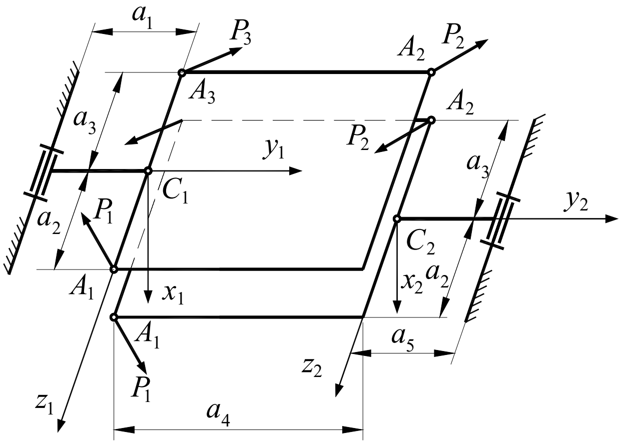

Figure 1. Example.

One get:

$$\mathbf{C_1O_1}=-a_1\mathbf{j},\ \mathbf{C_1O_1}\times\mathbf{i_1}=-a_1\mathbf{k},$$
$$\mathbf{C_1O_1}\times\mathbf{j_1}=\mathbf{0},\ \mathbf{C_1O_1}\times\mathbf{k_1}=-a_1\mathbf{i},\tag{19}$$

$$\mathbf{C_2O_2}=a_5\mathbf{j},\ \mathbf{C_2O_2}\times\mathbf{i_2}=-a_5\mathbf{k},$$
$$\mathbf{C_2O_2}\times\mathbf{j_2}=\mathbf{0},\ \mathbf{C_2O_2}\times\mathbf{k_2}=a_5\mathbf{i},\tag{20}$$

$$[\mathbf{S_1}]=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&-a_1&1&0\\0&0&0&0&1\\a_1&0&0&0&0\end{bmatrix},\ [\mathbf{Q_1}]=\begin{bmatrix}0\\0\\1\\-a_1\\0\\0\end{bmatrix},\tag{21}$$

$$[\mathbf{S_2}]=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&a_5&1&0\\0&0&0&0&1\\-a_5&0&0&0&0\end{bmatrix},\ [\mathbf{Q_2}]=\begin{bmatrix}0\\0\\1\\a_5\\0\\0\end{bmatrix},\tag{22}$$

$$\mathbf{C_1A_1}=a_2\mathbf{k},$$
$$\mathbf{C_1A_1}\times(\alpha_1\mathbf{i}+\beta_1\mathbf{j}+\gamma_1\mathbf{k})=a_2\alpha_1\mathbf{j}-a_2\beta_1\mathbf{i},\tag{23}$$

$$\mathbf{C_1A_2}=a_4\mathbf{j}-a_3\mathbf{k},$$
$$\mathbf{C_1A_2}\times(\alpha_2\mathbf{i}+\beta_2\mathbf{j}+\gamma_2\mathbf{k})$$
$$=(a_3\beta_2+a_4\gamma_2)\mathbf{i}-a_3\alpha_2\mathbf{j}-a_4\alpha_2\mathbf{k},\tag{24}$$

$$\mathbf{C_1A_3}=-a_3\mathbf{k},$$
$$\mathbf{C_1A_3}\times(\alpha_3\mathbf{i}+\beta_3\mathbf{j}+\gamma_3\mathbf{k})=-a_3\alpha_3\mathbf{j}+a_3\beta_2\mathbf{i},\tag{25}$$

$$\mathbf{C_2A_1}=-a_4\mathbf{j}+a_2\mathbf{i},$$
$$\mathbf{C_2A_1}\times(\alpha_1\mathbf{i}+\beta_1\mathbf{j}+\gamma_1\mathbf{k})$$
$$=-a_4\gamma_1\mathbf{i}-a_2\beta_1\mathbf{j}+(a_4\alpha_1+a_2\beta_1)\mathbf{k},\tag{26}$$

$$\mathbf{C_2A_2}=-a_3\mathbf{k},$$
$$\mathbf{C_2A_2}\times(\alpha_2\mathbf{i}+\beta_2\mathbf{j}+\gamma_2\mathbf{k})=-a_3\mathbf{j}+a_3\mathbf{i},\tag{27}$$

$$\mathbf{C_3A_3}=-a_4\mathbf{j}-a_3k,$$
$$\mathbf{C_3A_3}\times(\alpha_3\mathbf{i}+\beta_3\mathbf{j}+\gamma_3\mathbf{k})$$
$$=(a_3\beta_3-a_4\gamma_3)\mathbf{i}-a_3\alpha_3\mathbf{j}+a_4\alpha_3\mathbf{k},\tag{28}$$

$$[\mathbf{U_1}]=\begin{bmatrix}\alpha_1&\alpha_2&\alpha_3\\\beta_1&\beta_2&\beta_3\\\gamma_1&\gamma_2&\gamma_3\\A_1&A_2&A_3\\B_1&B_2&B_3\\C_1&C_2&C_3\end{bmatrix},$$
$$[\mathbf{U_2}]=\begin{bmatrix}\alpha_1&\alpha_2&\alpha_3\\\beta_1&\beta_2&\beta_3\\\gamma_1&\gamma_2&\gamma_3\\D_1&D_2&D_3\\E_1&E_2&E_3\\F_1&F_2&F_3\end{bmatrix},\tag{29}$$

$$\{\mathbf{v}^{0(1)}\}=[0\quad 0\quad -\omega_{10}\quad a_1\omega_{10}\quad 0\quad 0]^{\mathrm{T}},$$
$$\{\mathbf{v}^{0(2)}\}=[0\quad 0\quad -\omega_{20}\quad -a_5\omega_{20}\quad 0\quad 0]^{\mathrm{T}},\tag{30}$$

$$\{\mathbf{v}_{1n}^{(0)}\}=[\mathbf{U_1}]^{\mathrm{T}}[\boldsymbol{\eta}]\{\mathbf{v}^{0(1)}\}=\begin{bmatrix}a_1\omega_{10}\alpha_1-\omega_{10}C_1\\a_1\omega_{10}\alpha_2-\omega_{10}C_2\\a_1\omega_{10}\alpha_3-\omega_{10}C_3\end{bmatrix},$$
$$\{\mathbf{v}_{2n}^{0}\}=[\mathbf{U_2}]^{\mathrm{T}}[\boldsymbol{\eta}]\{\mathbf{v}^{0(2)}\}$$
$$=\begin{bmatrix}-a_5\omega_{20}\alpha_1-\omega_{20}F_1\\-a_5\omega_{20}\alpha_2-\omega_{20}F_2\\-a_5\omega_{20}\alpha_3-\omega_{20}F_3\end{bmatrix},\tag{31}$$

$$[\mathbf{M}_i]=\begin{bmatrix}0&0&0&m_i&0&0\\0&0&0&0&m_i&0\\0&0&0&0&0&m_i\\J_{x_i}&0&0&0&0&0\\0&J_{y_i}&0&0&0&0\\0&0&J_{z_i}&0&0&0\end{bmatrix},\quad i=\overline{1,2},\tag{32}$$

$$[\mathbf{M}_{1red}]=[\mathbf{Q_1}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_1][\mathbf{Q_1}]=m_1a_1^2+J_{z_1},$$
$$[\mathbf{M}_{2red}]=[\mathbf{Q_2}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_2][\mathbf{Q_2}]=m_2a_{51}^2+J_{z_2},\tag{33}$$

$$[\mathbf{G_1}]=[\mathbf{U_1}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q_1}][\mathbf{M}_{1red}]^{-1}[\mathbf{Q_1}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U_1}]=\frac{1}{m_1a_1^2+J_{z_1}}\begin{bmatrix}\lambda_{11}&\lambda_{12}&\lambda_{13}\\\lambda_{21}&\lambda_{22}&\lambda_{23}\\\lambda_{31}&\lambda_{32}&\lambda_{33}\end{bmatrix},$$
$$[\mathbf{G_2}]=[\mathbf{U_2}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q_2}][\mathbf{M}_{2red}]^{-1}[\mathbf{Q_2}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U_2}]=\frac{1}{m_2a_5^2+J_{z_2}}\begin{bmatrix}\nu_{11}&\nu_{12}&\nu_{13}\\\nu_{21}&\nu_{22}&\nu_{23}\\\nu_{31}&\nu_{32}&\nu_{33}\end{bmatrix},\tag{34}$$

$$\{\mathbf{v}_{12n}^0\}=\{\mathbf{v}_n^{0(1)}\}-\{\mathbf{v}_n^{0(2)}\}=[\delta_1\quad\delta_2\quad\delta_3]^{\mathrm{T}}\tag{35}$$

$$[\mathbf{G_1}]+[\mathbf{G_2}]=\begin{bmatrix}g_{11}&g_{12}&g_{13}\\g_{21}&g_{22}&g_{23}\\g_{31}&g_{32}&g_{33}\end{bmatrix},$$
$$[[\mathbf{G_1}]+[\mathbf{G_2}]]^{-1}=\begin{bmatrix}h_{11}&h_{12}&h_{13}\\h_{21}&h_{22}&h_{23}\\h_{31}&h_{32}&h_{33}\end{bmatrix},\tag{36}$$

$$\{\mathbf{P}\}=[[\mathbf{G}_1]+[\mathbf{G}_2]]^{-1}[[\mathbf{I}]+[\mathbf{K}]]\{\mathbf{v}_{12n}^0\}=[p_1\quad p_2\quad p_3]^{\mathrm{T}}.\tag{37}$$

For the first linkage one obtains

$$[\mathbf{Q_1}][\mathbf{M}_{1red}]^{-1}[\mathbf{Q_1}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U_1}]=\frac{1}{m_1a_1^2+J_{z_1}}\begin{bmatrix}0&0&0\\0&0&0\\C_1-a_1\alpha_1&C_2-a_1\alpha_2&C_3-a_1\alpha_3\\-a_1C_1+a_1^2\alpha_1&-a_1C_2+a_1^2\alpha_2&-a_1C_3+a_1^2\alpha_3\\0&0&0\\0&0&0\end{bmatrix},\tag{38}$$

$$[\mathbf{S_1}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M_1}]^{-1}[\mathbf{U_1}]\{\mathbf{P}\}=[\varepsilon_1\quad\varepsilon_2\quad\varepsilon_3\quad\varepsilon_4\quad\varepsilon_5]^{\mathrm{T}},\tag{39}$$

$$[[\mathbf{S_1}]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M_1}]^{-1}[\mathbf{S_1}]]^{-1}=\begin{bmatrix}L_{11}&0&0&0&0\\0&L_{22}&0&0&0\\0&0&L_{33}&L_{34}&0\\0&0&L_{43}&L_{44}&0\\0&0&0&0&L_{55}\end{bmatrix},\tag{40}$$

$$\{\xi_1\}=[[\mathbf{S}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_1]^{-1}[\mathbf{S}_1]]^{-1}[\mathbf{S}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_1]^{-1}[\mathbf{U}_1]\{\mathbf{P}\}=\begin{bmatrix}L_{11}\varepsilon_1\\L_{22}\varepsilon_2\\L_{33}\varepsilon_3+L_{34}\varepsilon_4\\L_{43}\varepsilon_3+L_{44}\varepsilon_4\\L_{55}\varepsilon_5\end{bmatrix}.\tag{41}$$

Similar results may be obtained for the second linkage.

One may observe that the matrix $[\mathbf{G_1}]+[\mathbf{G_2}]$ is not invertible any time. For instance, if the direction of the impulse $\mathbf{P_1}$ is $\mathbf{i}+\mathbf{j}$, the direction of the impulse $\mathbf{P_2}$ is $\mathbf{j}+\mathbf{k}$, the direction of the impulse $\mathbf{P_3}$ is $\mathbf{i}+\mathbf{k}$, and if $a_1=a$, $a_2=a$, $a_3=a$, $a_4=2a$, $a_5=a$, then

$$[\mathbf{G_1}]=\frac{1}{m_1a^2+J_{z_1}}\begin{bmatrix}a^2&0&a^2\\0&0&0\\a^2&0&a^2\end{bmatrix}.$$
$$[\mathbf{G_2}]=\frac{1}{m_2a^2+J_{z_2}}\begin{bmatrix}9a^2&0&9a^2\\0&0&0\\9a^2&0&9a^2\end{bmatrix}.\tag{42}$$

4. CONCLUSIONS

In this paper we discuss the simultaneous multi-points collision of two rigid bodies with constraints. The collision takes place at three different points. The particular case of the example given in the paper cannot be solved because it leads to a non-invertible matrix.

Based on the theory of screws, one may give the conditions in which the problem can be completely solved.

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