
Analytical Solution of Dynamic Analysis of Cracked Euler–Bernoulli Beam with Elastic Boundary Condition By G.F.M

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Abstract: - The recognition of behavior of the cracked beam causes to find out how can use all capability of beams. Existence crack during the length of the beam makes discontinuity on the beam and leads to reduce local stiffness. This paper presents the dynamic solution of the cracked Euler–Bernoulli beam by Green Function Method (G.F.M). Green function is exhibited for the Euler–Bernoulli beam with various boundary conditions. Also, discontinuity is modeled by rotational spring in this paper. The effects of crack in different locations and depths of cracks with considering various boundary conditions are assessed. In addition, the influence of crack on natural frequency is studied. Finally, several examples are presented to compare the effect of boundary conditions on the dynamic response of the weakened Euler–Bernoulli beam.

Keywords: Euler–Bernoulli beam – crack - dynamic analysis - Green Function Method.

1. INTRODUCTION

The recognition of behavior of the cracked beam causes to find out how can use all capability of beams. Many researchers are studied the dynamic response of the cracked beam due to the potential of the failure in the cracked beams. The multi-beams system accompaniment with transverse crack is studied by Saavedra and Cuitino [1]. The strain energy density function used to modelling the flexibility which has been created by crack. Yokoyama and Chen investigated on uniform cracked beam by using the line-spring model [2]. The natural frequency and mode shape are determined at different crack location and crack depth. Free vibration and buckling analysis of cracked beam are presented by Yang and Chen [3]. Vibrational characteristics, especially critical buckling load is obtained for cracked beam with various boundary condition. The dynamics behavior of Timoshenko cracked beam is analyzed using combination of finite element and component mode methods by Kisa et al. [4].

Dharmaraju et al investigated a method to reduce the termination of gauging the degrees of freedom of beams by the force response measurements [5]. The crack flexibility coefficient and the crack depth are anticipated using general identification algorithm method. A continuous bilinear model is used to obtaining the displacement of the cantilever cracked beam [6]. The Hamilton principle is used to solve governing equation. A crack beam, which subjected to axial and transverse loads is studied by Cicirello and Palmeri [7]. The static analysis is applied to determining the behavior of the cracked Euler–Bernoulli beam. Andreaus et al. applied a nonlinear

approach to determine dynamic response of the cantilever cracked beam [8]. Also, the method based on nonlinear output frequency response functions is investigated on cracked structures by Peng et al. [9]. The research is demonstrated that this approach are sensitive against of cracks. Ariaei et al. are studied a cracked beam under moving mass [10]. In this paper, the influences of load location and velocity on dynamic response of cracked beam are obtained using discrete element method. A non-uniform multiple cracked beam based on energy method is presented by Mazanoglu et al. [11].

Khorram et al used two approaches based on wavelet to recognize deflection of the cracked beam that is subjected to a moving load [12]. Praisach et al analyzed the damaged cantilever beam and obtained the natural frequency by the finite element method [13]. Mehrjoo et al presented a crack detection method based on genetic algorithms for identifying the position of cracks [14]. The Euler–Bernoulli and the Timoshenko nano-beams are studied to obtain and compare the characteristics of the cracked nano-beam such as shear deformation, mode number by Hosseini-Hashemi et al [15]. Nan Wu exhibited the forced vibration of the cracked beam [16]. For this purpose, the iteration numerical method is employed to solve the equation of the forced vibration cracked beam. The dynamic response of the cracked bridge beam subjected to the earthquake effects and moving vehicles is studied by Nguyen [17]. Chioncel et al employed the unique method to detect damage position on the beam [18]. A method based on tomography used to identify crack location. Alkassar studied the clamped beam with a crack in different location and depth [19]. Gillich et al used a method

based on natural frequency changes to find a crack location in composite structures. In this study, the natural frequency of the damaged and the undamaged sandwich beam are obtained and compared with experimental results [20].

In previous studies on the dynamic response of the cracked beams, only the beam with typical boundary condition analyzed. On the other hand, the solution from previous researchers cannot be generalized to the general boundary conditions. The primary objective of this paper is to present accurate vibration frequencies for the cracked beam with restrained against rotation and translation at its edge. An accurate solution in closed forms yields where the vibration expression for the cracked beam to be written in a simple formation. Hence, the computation becomes more efficient. At the same time, through the use of the presented method, the boundary conditions are embedded in the solution functions of the corresponding cracked beam. Therefore, the object of this paper is:

- To present a simple and practical analytical technique for determining the free vibration of the weakened Euler–Bernoulli beam with elastically restrained against rotation and translation.
- To present accurate and precise solutions in closed form solutions

2. GREEN FUNCTION FOR EULER-BERNOULLI BEAMS

In this paper, the creaked Euler–Bernoulli beam with elastic boundary condition are assumed. The governing differential equation on the Euler–Bernoulli beam is presented as [21]:

$$EI \frac{\partial^4 W}{\partial x^4} + c_i \frac{\partial^5 W}{\partial x^4 \partial t} + c_e \frac{\partial W}{\partial t} + \mu \frac{\partial^2 W}{\partial t^2} - q = 0 \quad (1)$$

where EI and μ are the flexural rigidity and the mass per unite length of the beam, respectively. W is the deflection of the beam and q is the applied load on beam. The constants c_e and c_i are the external and the internal damping effect. In order to remove the variety of time, a harmonic load $q = q(x) \text{Exp}(i \Omega t)$ is considered and by using $W = W(x) \text{Exp}(i \Omega t)$, equation (1) can be given by:

$$\left(1 + \frac{i \Omega c_i}{EI}\right) \frac{\partial^4 W}{\partial x^4} + \frac{i \Omega c_e - \mu \Omega^2}{EI} W(x) = \frac{q(x)}{EI} \quad (2)$$

where Ω is the circular frequency. By using of Green's function, Eq. (2) can be rewrite:

$$\left(1 + \frac{i \Omega c_i}{EI}\right) \frac{\partial^4 G}{\partial x^4} + \frac{i \Omega c_e - \mu \Omega^2}{EI} G(x) = \frac{\delta(x - x_0)}{EI} \quad (3)$$

where $\delta(\cdot)$ is Dirac delta function. By taking Laplace transform in Eq. (3) the following expressions are obtained [22]:

$$G(x; x_0) = \frac{-H(x - x_0)}{EI(1 + \frac{i \Omega c_i}{EI})} \phi(x - x_0) + \sum_{i=1}^4 \phi_i(x) \frac{\partial^{4-i} W(0)}{\partial x^{4-i}} \quad (4)$$

where $H(\cdot)$ is the Heaviside function. $\sum_{i=1}^4 \frac{\partial^{4-i} W(0)}{\partial x^{4-i}}$ are the boundary conditions at the edge beam. $\phi_i (i=1, 2, \dots, 4)$ can be determined from following equation:

$$\phi_i = \sum_{n=1}^4 R_n(x) S_n^{i-1} \quad (5)$$

In order to computing, $\phi_i (i=1, 2, \dots, 4)$, the following relation is considered:

$$\left(1 + \frac{i \Omega c_i}{EI}\right) S^4 + \frac{i \Omega c_e - \mu \Omega^2}{EI} = 0 \quad (6)$$

Solving Eq. (6) yields the following expressions:

$$S_{1,2} = \pm \sqrt[4]{\frac{(\mu \Omega^2 - i \Omega c_e)}{EI + i \Omega c_i}} \quad (7)$$

$$S_{3,4} = \pm i \sqrt[4]{\frac{(\mu \Omega^2 - i \Omega c_e)}{EI + i \Omega c_i}}$$

Thus, using Eq. (7), R_i becomes:

$$R_1(x) = \frac{e^{S_1 x}}{4S^3} \quad R_2(x) = -\frac{e^{S_2 x}}{4S^3} \quad (8)$$

$$R_3(x) = i \frac{e^{S_3 x}}{4S^3} \quad R_4(x) = -i \frac{e^{S_4 x}}{4S^3}$$

where, $S = \sqrt[4]{\frac{(\mu \Omega^2 - i \Omega c_e)}{EI + i \Omega c_i}}$.

3. CRACKED SECTION MODELING

In this paper, the massless torsional spring used to model cracked section of the beam. The crack manufactures two segments in the beam due to the

discontinuity. Therefore, the discontinuity in the slope of the beam deflection can be written as:

$$\Delta y = \frac{M}{k_c} \quad (9)$$

where Δy illustrates the slope, M is the bending moment that transmitted by the cracked section and k_c is the stiffness of the massless torsional spring. k_c can be expressed as:

$$k_c = \frac{EI}{(h)(D_{h'})} \quad (10)$$

where h is height of the beam; h' is the ratio of the depth of the crack (h_c) to the height of the beam (h)

($h' = h_c/h$) and $D_{h'}$ is the non-dimensional constant. The non-dimensional constant is presented using fracture mechanics. Several researchers are presented the different values of $D_{h'}$ for the case of rectangular section, which represented in Table 1 and plotted in Figures 1 and 2. As regards, the models of 1 to 4 are proportional to h'^2 and model 5 to h' , the charts of $D_{h'} - h'$ are plotted for five models. The charts shows negligible differences when the placed in $h' \in [0-0.6]$. Figure 1 Shows all five models work effectively. As showed in Figure 2, model 2 has the minimum and model 3 has the maximum value of the non-dimensional in $h' \in [0.6-0.8]$.

Table 1. Different value of $D_{h'}$ for the case of rectangular section

Model	Researchers	$D_{h'}$
1	Rizos et al [23]	$5.346h'^2 \left(1.86 - 3.95h' + 16.375h'^2 - 37.226h'^3 + 7.81h'^4 - 126.9h'^5 + 172h'^6 - 143.97h'^7 + 66.56h'^8 \right)$
2	Ostachowicz et al [24]	$6\pi h'^2 \left(0.6384 - 1.035h' + 3.720h'^2 - 5.1773h'^3 + 7.553h'^4 - 7.332h'^5 + 2.4909h'^6 \right)$
3	Fernandez-Saez et al [25]	$2 \left(\frac{h'}{1-h'} \right)^2 \left(5.93 - 19.69h' + 37.14h'^2 - 35.84h'^3 + 13.12h'^4 \right)$
4	Chondros et al [26]	$6\pi(1-\nu^2)h'^2(0.6272 - 1.04533h' + 4.5948h'^2 - 9.9736h'^3 + 20.2948h'^4 - 33.0351h'^5 + 47.1063h'^6 - 40.755h'^7 + 19.6h'^8)$
5	Bilello [27]	$\frac{h'(2-h')}{0.9(1-h')^2}$

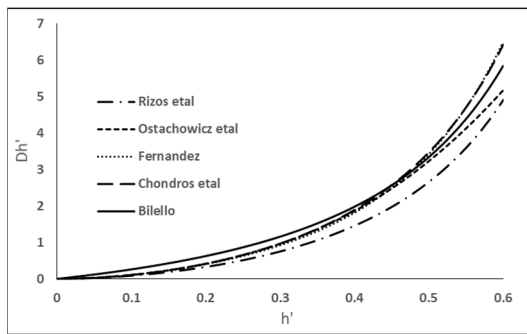


Figure 1. The non-dimensional constant spring model at $h'[0-0.6]$

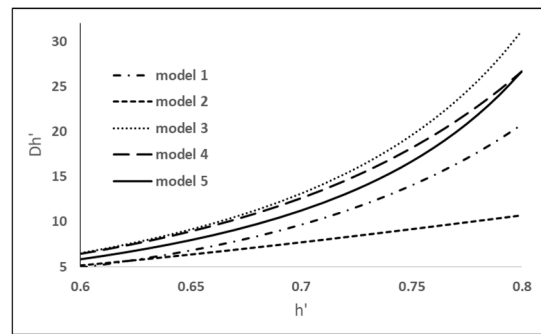


Figure 2. The non-dimensional constant spring model at $h'[0.6-0.8]$

4. GREEN FUNCTION FOR CRACKED BEAM

Consider a cracked Euler-Bernoulli beam where it is restrained against translation and rotation at its edge as shown in Figure 3. The length and height of

the beam are, respectively, L and h , and it has a crack depth if h_c , at a distance L_1 from the left support. The cracked beam subjected to a time harmonic concentrated load placed at x_0 . In order to obtain Green function for cracked beam, the cracked beam separate into two segments ($x_1 \in [0, L_1]$ and

$x_2 \in (L_1, L]$). For each segment employed the expanded state of Eq. (4):

$$G_1(x_1; x_{10}) = -\frac{H(x_1 - x_{10})}{EI \left(1 + i \frac{\Omega c_i}{EI}\right)} \phi_1(x_1 - x_{10}) + \lambda_{10} \phi_4(x_1) + \lambda_{11} \phi_3(x_1) + w_1'(0) \phi_2(x_1) + w_1''(0) \phi_1(x_1) \quad (11)$$

$$G_2(x_2; x_{20}) = -\frac{H(x_2 - x_{20})}{EI \left(1 + i \frac{\Omega c_i}{EI}\right)} \phi_1(x_2 - x_{20}) + \lambda_{20} \phi_4(x_2) + \lambda_{21} \phi_3(x_2) + w_2'(0) \phi_2(x_2) + w_2''(0) \phi_1(x_2)$$

where x_{10} and x_{20} denotes the coordinate of position of the external load in the local coordinate system. As respects, the boundary conditions in Figure 3 are elastic, the following coefficient used to produce elastic boundary condition in Eq. (11):

$$\lambda_{10} = \frac{-EI}{k_{LT}} \times w_1''(0) \quad \lambda_{11} = \frac{EI}{k_{LR}} \times w_1'(0) \quad (12)$$

$$\lambda_{20} = \frac{-EI}{k_{RT}} \times w_2''(0) \quad \lambda_{21} = \frac{EI}{k_{RR}} \times w_2'(0)$$

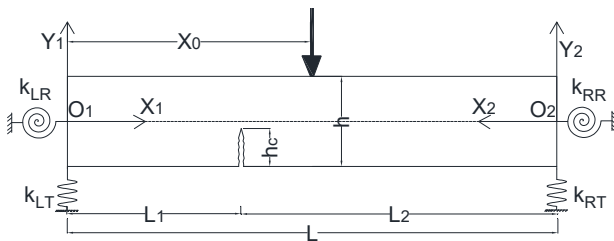


Figure 3. Cracked beam with elastic boundary condition

where k_{LT} , k_{RT} , k_{LR} and k_{RR} are transferred and rotational stiffness at the left and right support. According to continuity equation at the cracked section, the following relations in the local coordinate systems can be expressed [4]:

$$G_1 = G_2 \quad G_1' = G_2' \quad (13)$$

$$G_1'' = -G_2'' \quad G_1' = -\frac{EIG_2'' - G_2'}{k_c}$$

By substitution Eq. (11) in Eq. (13), the following matrix can be obtained:

$$\begin{bmatrix} \psi_1(x_1) & \psi_2(x_1) & -\psi_3(x_2) & \psi_4(x_2) \\ \psi_1'(x_1) & \psi_2'(x_1) & -\psi_3'(x_2) & \psi_4'(x_2) \\ \psi_1''(x_1) & \psi_2''(x_1) & \psi_3''(x_2) & -\psi_4''(x_2) \\ \psi_1'''(x_1) & \psi_2'''(x_1) & \gamma_3(x_2) & \gamma_4(x_2) \end{bmatrix} \quad (14)$$

where:

$$\psi_1(x_1) = \frac{EI}{k_{LR}} \phi_3(x_1) + \phi_2(x_1)$$

$$\psi_2(x_1) = -\frac{EI}{k_{LT}} \phi_4(x_1) + \phi_1(x_1)$$

$$\psi_3(x_2) = \frac{EI}{k_{RR}} \phi_3(x_2) + \phi_2(x_2)$$

$$\psi_4(x_2) = \frac{EI}{k_{LR}} \phi_4(x_2) - \phi_1(x_2)$$

$$\gamma_3(x_2) = \frac{(EI)^2}{k_c k_{RR}} \phi_3''(x_2) + \frac{EI}{k_{RR}} \phi_3'(x_2) + \frac{EI}{k_c} \phi_2''(x_2) + \phi_2(x_2)$$

$$\gamma_4(x_2) = -\frac{(EI)^2}{k_c k_{RT}} \phi_4''(x_2) - \frac{EI}{k_{RT}} \phi_4'(x_2) + \frac{EI}{k_c} \phi_1''(x_2) + \phi_1(x_2)$$

By calculation the determinant of Eq. (14) the natural frequency of elastic cracked beam is available.

5. NUMERICAL EXAMPLES

The simply supported beam is considered based on Zhao et al. [28] data's to evaluate present approach and dynamic analysis is applied to obtain the behavior of the cracked beam in various depth (see Figure 4).

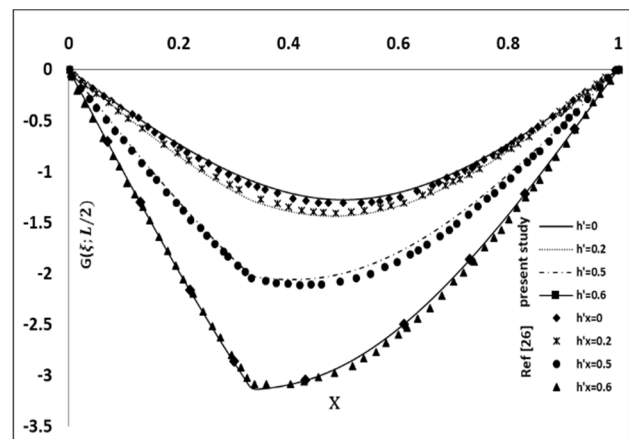


Figure 4. Variations of dynamic response of the simply supported cracked beam with the different crack depth

In order to the modelling of simply supported beam, as illustrated in Figure 3, k_{LT} and k_{RT} should be large number and k_{LR} and k_{RR} is equal to zero. According to Figure 4, there are insignificant differences in comparison with Zhao et al results.

5.1. The effect of the crack position on the dynamic response of the cracked beam

A rectangular beam with following characteristics is considered:

$$L = 2.1m, \quad b = 0.25m, \quad h = 0.1m$$

$$E = 201 \times 10^9 N/m^2, \quad \rho = 7800 kg/m^3$$

In order to compare the effects of crack during the length of beam, the cracked beam according to location of crack is classified. The simply supported cracked beam, which crack is located at $0.1L, 0.2L, 0.3L, 0.4L$ and $0.5L$ of left support under a concentrated load at $x = L/2$ with $\Omega = 119 rad/s$ are studied.

The dynamic response of the simply supported cracked beam with different location of crack presented in Figure 5. As seen from the Figure 5, when the location of the crack is close to the beam support, (i.e. located at $0.1L$), the behavior of the weakened beam, with ascending of crack depth is insignificant. The deflection of the cracked beam increases by moving the crack to the middle of the span. Due to the presence of the load at $0.5L$, the crack effect is more evident in middle of the span. In cases which $h' \in [0-0.3]$, changes of $G(x; L/2)/W_{max}^s$ are small. It means that the influence of surface cracks on beam behavior is inconsiderable. The crack effects on the natural frequency of the simply supported cracked beam is exhibited in Table 2. According to Table 2, by increasing the value of h' , the frequency of the cracked beam decreases, especially, at the middle of the span. It should be noted here that the importance of crack location on natural frequency is considerable. In order to compare the achieved result of present method with Finite Element Method (FEM), the first three natural frequency of clamped cracked beam obtained based on ref [19] data's and the result has been presented in Table 3. According to Table 3, there are insignificant errors.

Table 2. Natural frequency in the simply supported cracked beam

Crack position		0.1L	0.2L	0.3L	0.4L	0.5L
Crack ratio	$h'=0$	327.96	327.96	327.96	327.96	327.96
	$h'=0.1$	327.572	326.56	325.324	324.337	323.963
	$h'=0.3$	326.242	321.847	316.671	312.706	311.244
	$h'=0.5$	323.044	311.001	297.979	288.819	285.605
	$h'=0.7$	311.847	278.012	249.417	232.811	227.501

5.2. Cracked beam with elastic boundary condition

A cracked beam with elastic boundary condition considered (see Figure 3). It is assumed that a crack (crack ratio $h' = 0.5$) is at $L/3$ of left support of beam. Dynamic behavior of the cracked beam with the elastic boundary condition is presented in Table 4. The values of K_{LR} and K_{RR} are assumed zero. According to Table 4, the maximum frequency of the beam occurs when the values of K_{LT} and K_{RT} are zero. By increasing the value of K_{LT} and K_{RT} , the value of natural frequency ascends, also the ratio of displacement at $L/3$ and $L/2$ enhances. In case which K_{LT} and K_{RT} equal 1000000, the beam

behaves such a simply support beam, it means the beam goes against transferring.

Now let us consider an elastic-free cracked beam. The assumption of crack depth and crack location is same as last example. Dynamic behavior of the cracked beam with the elastic-free boundary condition presented in Figure 6. The values of K_{RT} and K_{RR} are assumed zero. The value of K_{LT} and K_{LR} are the same (K) and the concentrated load exert at $L/2$. Due to absorbing, the entire applied load forces by ascending K_{LT} and K_{LR} , the value of $G(x; L/2)$ increases. In this case, the behavior of the cracked beam is such cantilever beam. By ascending stiffness in beam support, the beam is more capable to absorbing the force.

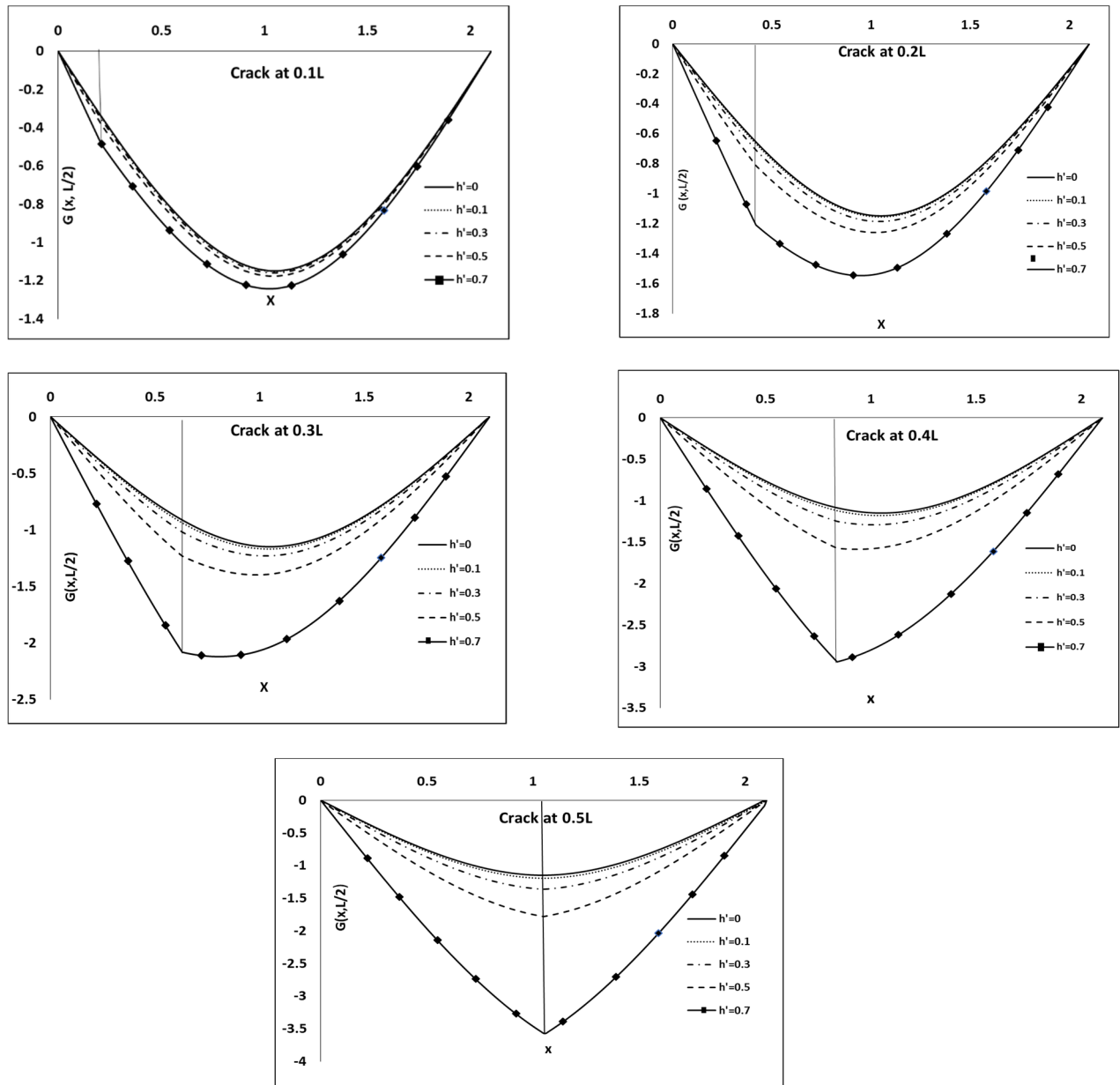


Figure 5. Variations of dynamic response of the simply supported cracked beam with the different crack depth

Table 3. Comparison of the three first natural frequency obtained by the Green Function method with the FEM

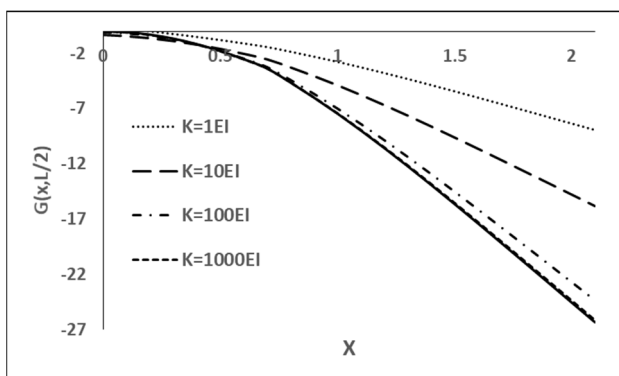
Crack ratio	Crack location	FEM result [19]			Present study			Error (%)		
		FNF	SNF	TNF	FNF	SNF	TNF	FNF	SNF	TNF
0.2	0.32	0.9995	0.9988	0.9987	0.9990	0.9978	0.9974	0.0540	0.1001	0.1302
	0.48	0.9994	0.9986	0.9985	0.9990	0.9978	0.9974	0.0430	0.0801	0.1102
0.4	0.16	0.9907	0.9965	0.9997	0.9883	0.9954	0.9997	0.2423	0.1104	0.0000
	0.64	0.9895	0.9960	0.9998	0.9883	0.9954	0.9997	0.1213	0.0602	0.0090
0.6	0.24	0.9880	0.9657	0.9940	0.9882	0.9658	0.9938	0.0202	0.0104	0.0201
	0.56	0.9892	0.9694	0.9947	0.9882	0.9658	0.9938	0.1011	0.3714	0.0905
0.8	0.24	0.9691	0.9040	0.9811	0.9651	0.8881	0.9769	0.4128	1.7588	0.4281
	0.64	0.9187	0.9643	0.9976	0.9021	0.9541	0.9972	1.8069	1.0578	0.0401

FNF, SNF and TNF is the first, second and third natural frequency, respectively

Table 4. Dynamic behavior of the cracked beam with the elastic boundary condition with a crack at $L/3$

k_{LT}	Response	k_{RT}							
		0	1	10	100	1000	10000	100000	1000000
0	$\Delta_{L/3}$	3.612	3.537	3.562	3.564	3.564	3.564	3.564	3.564
	$\Delta_{L/2}$	3.563	1.832	2.411	2.448	2.452	2.452	2.452	2.452
	Ω	664.136	691.426	408.768	469.509	475.634	476.24	476.303	476.309
1	$\Delta_{L/3}$	0.176	21.445	7.175	6.720	6.678	6.673	6.673	6.673
	$\Delta_{L/2}$	1.778	21.691	6.201	5.707	5.661	5.656	6.673	5.656
	Ω	190.247	130.651	154.357	156.333	156.524	156.543	156.545	156.546
10	$\Delta_{L/3}$	1.336	4.878	1.79	1.642	1.628	1.627	1.626	1.626
	$\Delta_{L/2}$	2.381	6.091	1.881	1.680	1.661	1.659	1.658	1.658
	Ω	385.504	155.267	252.055	267.444	268.976	269.129	269.144	269.146
100	$\Delta_{L/3}$	1.411	4.354	1.534	1.397	1.384	1.383	1.383	1.382
	$\Delta_{L/2}$	2.420	5.598	1.676	1.485	1.467	1.465	1.465	1.465
	Ω	437.924	157.354	268.847	289.409	291.491	291.700	291.721	291.723
1000	$\Delta_{L/3}$	1.418	4.306	1.510	1.374	1.361	1.360	1.359	1.359
	$\Delta_{L/2}$	2.424	5.552	1.657	1.467	1.449	1.447	1.447	1.447
	Ω	443.409	157.55	270.554	291.690	293.844	294.059	294.081	294.083
10000	$\Delta_{L/3}$	1.419	4.301	1.508	1.371	1.358	1.357	1.357	1.357
	$\Delta_{L/2}$	2.424	5.548	1.655	1.465	1.447	1.445	1.445	1.445
	Ω	443.957	157.576	270.725	291.920	294.080	294.296	294.317	294.319
100000	$\Delta_{L/3}$	1.419	4.300	1.507	1.371	1.358	1.357	1.357	1.357
	$\Delta_{L/2}$	2.424	5.547	1.655	1.465	1.447	1.445	1.445	1.445
	Ω	444.012	157.578	270.742	291.943	294.103	294.319	294.341	294.343
1000000	$\Delta_{L/3}$	1.419	4.300	1.507	1.371	1.358	1.357	1.357	1.357
	$\Delta_{L/2}$	2.424	5.547	1.655	1.465	1.447	1.445	1.445	1.445
	Ω	444.017	157.578	270.744	291.945	294.106	294.322	294.343	294.345

$$\Delta_{L/3} = \frac{G(x; L/3)}{W_{\max}^s}; \Delta_{L/2} = \frac{G(x; L/2)}{W_{\max}^s}; W_{\max}^s = \frac{L^3}{48EI}$$

**Figure 6.** Effects of support stiffness on cracked beam

6. CONCLUSIONS

In this study, the behavior of the cracked Euler-Bernoulli beam investigated. The effects of existence crack during the length of the beam on the displacement and the natural frequencies is studied. When the crack is close to the beam support, the effect of crack on the beam displacement is insignificant, but by moving crack to the middle of beam, the effect of crack is more evident. When the load and crack are in same position, the natural frequency is at its lowest value. The natural frequency of cracked beam compared and the insignificant errors is observed. The presented method is more

efficient and simplistic in comparison with the other methods because the method yields precise solutions in closed forms. In addition, the boundary conditions can be embedded in this method. Finally, the numerical examples are shown to illustrate the efficiency and simplicity of the new formulation base.

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