Gibbs-Appell Equations used in Finite Element Analysis of Elastic Materials with Voids having a Plane Motion

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Abstract: - The paper aims to use the Gibbs-Appell equations to obtain the evolution equations of a finite element (FE) to solve a problem that is becoming more and more current in the context of the accelerated use of composite materials in engineering. It is the problem of modeling materials with voids, which appear accidentally or are intentionally introduced into a material. In general, Lagrange's equations represent the most used mathematical tool for solving such cases. The complexity of current engineering applications, in which numerous parameters must be taken into account and numerous steps must be completed in the modeling activity, make some of the classical methods of Analytical Mechanics to be reconsidered, in order to take advantage of the particularities offered by them. This can reduce the number of procedures to follow in modeling the system and also reduce the amount of computation required (and costs). In this context the Gibbs-Appell method is used to finally obtain the equations of motion for a material with voids. For example, it can be noted that for a concentration of voids of 4% the stresses increase by 12%.

Keywords: - vibration, material with void, Gibbs-Appell equations, FEM, two dimensional FEs.

1. INTRODUCTION

The impressive industrial development of the last period has led to a continuous complexity of the engineering applications studied and the materials used. In fields such as aviation, the automotive industry, the naval industry, manufacturing, medicine, etc. A current defect encountered in the manufacture of plastic and composite materials is the existence of voids, which generally appear accidentally and with unfavorable effects on the resistance and functioning of the manufactured elements. However, these materials may contain voids accidentally created during production which, in certain situations, can significantly alter the material properties. Usually, these voids appear in the manufacturing process and represent defects, but they can also appear controlled for certain technical reasons. Obviously, such a problem aroused the interest of researchers in the phenomenon [1]. Until now, most researches are experimental, the complexity of the problem has determined that theoretical researches are few and far between. Thus, experiments were made to identify the mechanical properties of such materials. The number of parameters that must be considered in these studies is generally large [2,3]. The first attempts to develop numerical calculation methods are presented in [4-6]. The voids that appear in the studied materials are in most applications filled with air. Obviously, they can be filled with various other gases, determined by the manufacturing process. The appearance of voids is an undesirable thing, a defect. They have an influence on the mechanical properties of the studied materials, which can sometimes be significant. They can even cause the destruction of the material or the manufactured part and can reduce the life of the mechanical system in which they intervene. Some types of holes can change the isotropy of the material.

The development of theoretical models that make possible the numerical approach to the problems caused by voids is required.

Technical applications that lead to the appearance of voids are presented in various works such as [7,8]. The main problem created by voids is that, even in small concentrations, they can lead to significant variations in the mechanical properties of the material [9-11].

A calculation of strain and stress for different percentages of voids appearing in the material of the bar is presented in Table 1 [59]. It is found that voids have an influence on the deformations of the bar in the sense of their growth, stronger than linear.

Table 1. Displacements and stress for a bar with voids [59]

	Displacement[mm]	Stress [MPa]
Without voids	0.5126	22.995
1 % voids	0.5322	23.541
2 % voids	0.5524	24.213
3 % voids	0.5586	24.805
4 % voids	0.5723	25.323

The analysis of the results in this table shows us that a small percentage of voids in the material can cause a significant variation of the mechanical properties, manifested by the variation of strains and stresses.

The first studies in the field are by Cowin and Goodman [12]. This study led to applications in various fields, in the research of soils, rocks and human bones [13-16]. The difficulty of studying these systems is due to the complexity of the equations that appear. If all these problems are taken into account in the study of hollow materials, the finite element method (FEM) proves to be a powerful tool for their study [17].

The use of FEM has become a current practice in the analysis of the properties of bi- or multiphase composite materials [18-24]. The first researches regarding the application of FEM to bodies with voids are presented in [25] and developments of the method are made in [26,27]. The present work offers a model with finite elements for determining the mechanical behavior of a linear elastic body with voids, using alternative formulations from Analytical Mechanics. In the case of using Lagrange's equations to obtain the evolution equations, the main difficulty is the complexity of the analytical formulas. To avoid this problem, the authors propose the use of Gibbs-Appell equations and the use of the little-used notion of energy of accelerations. Applying these equations, using classical FEM assembly methods, with the corresponding boundary conditions, differential equations are obtained that describe the evolution of the system over time. Within this approach there is an important number of researchers who studied the problem using a simple one-dimensional finite elements [28-32], then more sophisticated finite elements [33-35]. In the case of all these studies, Lagrange's equations were preferred for system modeling [36-39], which is generally determined by their generality and the researchers' familiarity with these equations. Recent works have begun to reconsider other alternative methods of analytical mechanics. The Gibbs-Appel equations were considered by the authors for the study in the current work. The main advantage is the decrease in the number of necessary differentiation operations, which leads to a decrease in modeling and calculation time.

The method was used independently by Gibbs in 1879 [40] and by Appell in 1899 [41]. In essence, this method represents an application of Gauss' principle of minimum constraint. Of course, the same results are obtained as in the case of classical methods. The extension of this method to systems with gaps represents a natural extension of the method in the current context of technological development and powerful existing computational methods [42] . Applications of the method, which prove its usefulness, can be found in [43-46]. The main advantage in the case of applying this method is the reduced number of necessary operations. Currently, the method is being reconsidered by several researchers who used it for their studies [47-51]. Other results concerning the materials with voids and their dynamics can be found in [52-58]. In this research, the FEM is used together with a Gibbs-Appell method. The justification is to obtain computational advantages, and a shorter modeling time. Consequently, there will be lower costs for conducting the research and obtaining the results.

In the research, I presented the possibility of using a classical formalism from Analytical Mechanics which, for non-holonomous mechanical systems, can lead to advantages in terms of the number of operations that must be performed. The form of writing the equations of motion involves three phases of differentiations of the Lagrange function while the use of Gibbs-Appell involves only one set of differentiations of the scalar expression of energy. The advantage is obvious. The disadvantage is constituted by the lack of familiarity of the researcher with the energy of accelerations.

2. FINITE ELEMENT MODELING

2.1. Basic Kinematics

In the following the basic notations used in the Finite Element Analysis (FEA) will be presented [36]. These notations will be used in our presentation

of the Gibbs-Appell method. First of all, only one single FE will be studied. It is considered a local reference system participating to the rigid motion of the element of the system to which the finite element belongs. This FE has an elastic deformation. Every FE is referred to a local coordinate system and all these Fes will be refer to a global reference frame, in order to describe the motion of the whole mechanical system. It is denoted with $\bar{v}_0(v_{O1}, v_{O2})$ the velocity of the local reference frame and with $\bar{a}_0(a_{O1}, a_{O2})$ his acceleration. The local coordinate system has, for the considered element of the multibody system (MBS) the angular velocity $\bar{\omega} = \omega \bar{k}$ and angular acceleration $\bar{\varepsilon} = \varepsilon \bar{k}$. In the following considerations the indexes *L* (local) and *G* (global) show whether a size is expressed relative to the local (*L*) or global (*G*) reference frames, respectively (if it is necessary to understand the context). Considering a plane motion, the angle θ represent the rotation angle of the local reference system relative to the global fixed reference system. The rotation matrix [R] is:

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \qquad (1)$$

Considering this rotation matrix, an arbitrary vector has the components expressed in the global coordinate system:

$$\begin{aligned} A_1 &= a_1 \cos\theta - a_2 \sin\theta \quad ; \\ A_2 &= a_1 \sin\theta + a_2 \cos\theta \quad . \end{aligned} \qquad (2)$$

It will be denoted with $\bar{r}_O$ the vector of the origin of the local reference system, refered to the global reference frame. O_1 is the origin of the fixed global coordinate system. The components of the vector $\bar{r}_O$ are, (X_O, Y_O) and (x_O, y_O) respectively in the global/local reference system. Here $\bar{r}_M$ represents the position vector of the point M before deformation, with the components (X_M, Y_M) in the global coordinate system and (x_M, y_M) in the local reference frame and $\bar{r}_{M'}$ the same vector after deformation when becomes *M'* with the components $(X_{M'}, Y_{M'})$, respectively $(x_{M'}, y_{M'})$. These vectors are expressed considering the origin O_1. The position vectors of the points *M* considering the origin *O* of the local coordinate system are $\bar{r}$ (with the components (X, Y), respectively (x, y)).

The considered point *M* of a finite element becames *M'* after deformation and its coordinates are:

$$\begin{aligned} X_{M'} &= X_O + (x+u)\cos\theta - (y+v)\sin\theta \quad ; \\ Y_{M'} &= Y_O + (x+u)\sin\theta + (y+v)\cos\theta \, . \end{aligned} \qquad (3)$$

FEM approximates the displacement of a point *u* and *v* using the shape functions N_{1j}, N_{2j}, $j = \overline{1,p}$, by the following relations [22]:

$$\begin{aligned} u &= N_{1j}\delta_j \quad ; \quad j = \overline{1,p} \\ v &= N_{2j}\delta_j \quad ; \quad j = \overline{1,p} \end{aligned} \qquad (4)$$

where δ_1, δ_2, ..., δ_p are the independent coordinates that define the element and *p* is the number of these coordinates (see Fig.1). Using Eq.(4), Eq.(3) becomes:

$$\begin{aligned} X_{M'} &= X_O + \left(x + N_{1j}\delta_j\right)\cos\theta - \left(y + N_{2j}\delta_j\right)\sin\theta \quad ; \\ Y_{M'} &= Y_O + \left(x + N_{1j}\delta_j\right)\sin\theta + \left(y + N_{2j}\delta_j\right)\cos\theta \quad ; \\ & \qquad j = \overline{1,p}. \end{aligned} \qquad (5)$$

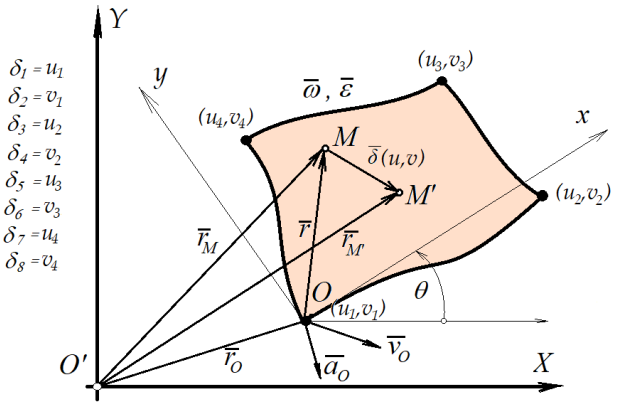

Figure 1. A plane finite element

So, the velocity vector of M' has the components:

$$\begin{aligned} \dot{X}_{M'} &= \dot{X}_O - \omega\left(x + N_{1j}\delta_j\right)\sin\theta \\ &+ \left(x + N_{1j}\dot{\delta}_j\right)\cos\theta - \omega\left(y + N_{2j}\delta_j\right)\cos\theta \\ &- \left(y + N_{2j}\dot{\delta}_j\right)\sin\theta \quad ; \end{aligned} \qquad (6)$$

$$\begin{aligned} \dot{Y}_{M'} &= \dot{Y}_O + \omega\left(x + N_{1j}\delta_j\right)\cos\theta \\ &+ \left(x + N_{1j}\dot{\delta}_j\right)\sin\theta - \omega\left(y + N_{2j}\delta_j\right)\sin\theta \\ &+ \left(y + N_{2j}\dot{\delta}_j\right)\cos\theta \quad ; \quad j = \overline{1,p} \quad . \end{aligned} \qquad (7)$$

The components of accelerations are:

$$\begin{aligned} \ddot{X}_{M'} &= \ddot{X}_O - \varepsilon\left(x + N_{1j}\delta_j\right)\sin\theta \\ &- \omega^2\left(x + N_{1j}\delta_j\right)\cos\theta - 2\omega\left(x + N_{1j}\dot{\delta}_j\right)\sin\theta \\ &+ \left(x + N_{1j}\ddot{\delta}_j\right)\cos\theta - \varepsilon\left(y + N_{2j}\delta_j\right)\cos\theta \\ &+ \omega^2\left(y + N_{2j}\delta_j\right)\sin\theta - 2\omega\left(y + N_{2j}\dot{\delta}_j\right)\cos\theta \\ &- \left(y + N_{2j}\ddot{\delta}_j\right)\sin\theta \quad ; \end{aligned} \qquad (8)$$

$$\begin{aligned} \ddot{Y}_{M',2} &= \ddot{Y}_O + \varepsilon\left(x + N_{1j}\delta_j\right)\cos\theta \\ &- \omega^2\left(x + N_{1j}\delta_j\right)\sin\theta + 2\omega\left(x + N_{1j}\dot{\delta}_j\right)\cos\theta \end{aligned}$$

$$+\left(x+N_{1j}\ddot{\delta}_j\right)\sin\theta-\varepsilon\left(y+N_{2j}\delta_j\right)\sin\theta$$
$$-\omega^2\left(y+N_{2j}\delta_j\right)\cos\theta-2\omega\left(y+N_{2j}\dot{\delta}_j\right)\sin\theta$$
$$+\left(y_2+N_{2j}\ddot{\delta}_j\right)\cos\theta \quad ; \quad j=\overline{1,p}. \tag{9}$$

Considering the local reference frame the previous relations become, successively:

$$\begin{aligned} x_{M'} &= x_O + x + N_{1j}\delta_j \quad ; \\ y_{M'} &= y_O + y + N_{2j}\delta_j \quad ; \quad j=\overline{1,p}. \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{x}_{M'} &= \dot{x}_O - \omega y - \omega N_{2r}\delta_r + N_{1r}\dot{\delta}_r \quad ; \\ \dot{y}_{M'} &= \dot{y}_O + \omega x + \omega N_{1r}\delta_r + N_{2r}\dot{\delta}_r \quad ; r=\overline{1,p}. \end{aligned} \tag{11}$$

2.2. Voids

In the case of a material with voids the density $\rho(x,y,z)$ of the body can be expressed by:

$$\rho(x,y,z)=v(x,y,z)\,\gamma(x,y,z) \tag{12}$$

Here, $\gamma(x,y,z)$ is the mass density of the solid if the voids do not exist, and $v(x,y,z)$ represents the volume fraction of the existing material, ($0<\nu\le 1$). At an initial moment:

$$\rho_o(x,y,z)=\nu_o(x,y,z)\,\gamma_o(x,y,z) \ . \tag{13}$$

The distribution of the percentage of voids can be written as [60]:

$$v_L=\left[N_v\right]\{\delta\}_L \ . \tag{14}$$

Here, the quantity ν_L is scalar, and the shape matrix $\left[N_\nu\right]$ is a line matrix. There is the relation [60]:

$$\dot{v}_L=\left[N_\nu\right]\left\{\dot{\delta}\right\}_L \ , \tag{15}$$

and the kinetic energy, due to the existence of the voids:

$$E_{cv}=\frac{1}{2}\int_V \rho_0\kappa_\nu\dot{v}^2_{M'G}dV \ . \tag{16}$$

It is denoted with κ_ν the so called "equilibrated inertia". It is used the notation:

$$\left[m_\nu\right]=\int_V\left[N_\nu\right]^T\left[N_\nu\right]\kappa\,\rho_o dV \ . \tag{17}$$

The term of potential energy due to the voids is:

$$E_{pv}=\frac{1}{2}\{\delta\}_L^T\left[k_v\right]\{\delta\}_L \ . \tag{18}$$

2.2. Kinetic Energy

The expression for the kinetic energy is:

$$E_C=\frac{1}{2}\int_V\rho\left[(\dot{x}_{M'})^2+(\dot{y}_{M'})^2\right]dV$$
$$=\frac{1}{2}\int_V\rho\left(\dot{x}_O-\omega x_2-\omega N_{2r}\delta_r+N_{1r}\dot{\delta}_r\right)^2dV$$
$$+\frac{1}{2}\int_V\rho\left(\dot{x}_{O,1}+\omega x_1+\omega N_{1r}\delta_r+N_{2r}\dot{\delta}_r\right)^2dV. \tag{19}$$

or:

$$E_C=\frac{1}{2}\left(\dot{x}^2_{O,1}+\dot{x}^2_{O,2}\right)\int_V\rho dV+\frac{1}{2}\omega^2\int_V\rho\left(x_1^2+x_2^2\right)dV$$
$$+\frac{1}{2}\omega^2\delta_t\delta_r\int_V\rho\left(N_{2r}N_{2t}+N_{1r}N_{1t}\right)dV$$
$$+\frac{1}{2}\dot{\delta}_r\dot{\delta}_t\int_V\rho\left(N_{1r}N_{1t}+N_{2r}N_{2t}\right)dV$$
$$-\omega\left(\dot{x}_{O,1}\int_V\rho x_1dV-\omega\dot{x}_{O,2}\int_V\rho x_2dV\right)$$
$$-\omega\delta_r\left(\dot{x}_{O,1}\int_V\rho N_{2r}dV+\dot{x}_{O,2}\int_V\rho N_{1r}dV\right)$$
$$+\left(\dot{x}_{O,1}\int_V\rho N_{1r}dV+y\dot{x}_{O,2}\,\dot{\delta}_r\int_V\rho N_{2r}dV\right)\dot{\delta}_r$$
$$+\omega^2\delta_r\left(\int_V\rho N_{2r}x_2\,dV+\int_V\rho N_{1r}x_1\,dV\right)$$
$$-\omega\dot{\delta}_r\left(\int_V\rho x_2N_{1r}dV-\int_V\rho x_1N_{2r}dV\right)$$
$$-\omega\delta_r\dot{\delta}_t\left(\int_V\rho N_{2r}N_{1t}dV-\int_V\rho N_{1r}N_{2t}dV\right) \tag{20}$$

Using the notations:

$$m=\int_V\rho dV \quad ; \quad J_O=\int_V\rho\left(x_1^2+x_2^2\right)dV \quad ;$$
$$m_{rt}=\int_V\rho N_{kr}N_{kt}dV; \quad m_{ij,rt}=\int_V\rho N_{ir}N_{jt}dV \quad ;$$
$$S_1=\int_V\rho x_1dV \quad ; \quad S_2=\int_V\rho x_2dV \quad ;$$
$$m^I_{O,kr}=\int_V\rho N_{kr}dV \quad ; \quad m_{1,mr}=\int_V\rho\,xN_{mr}dV \quad ;$$
$$m_{2,mr}=\int_V\rho yN_{mr}dV \quad ; \tag{21}$$

the Eq. (13) becomes:

$$E_c=\frac{1}{2}m\left(\dot{x}^2_{O,1}+\dot{x}^2_{O,1}\right)+\frac{1}{2}\omega^2J_O$$
$$+\frac{1}{2}\omega^2\delta_t\delta_rm_{rt}+\frac{1}{2}\dot{\delta}_r\dot{\delta}_tm_{rt}$$
$$-\omega\left(\dot{x}_{O,1}S_2-\dot{x}_{O,2}S_2\right)-\omega\delta_r\left(\dot{x}_{O,1}m^I_{O,2r}+\dot{x}_{O,2}m^I_{O,1r}\right)$$

$$+\left(\dot{x}_{O,1}m^I_{O,1r}+\dot{x}_{O,2}\dot{\delta}_r m^I_{O,2r}\right)\dot{\delta}_r+\omega^2\delta_r\left(m_{2,2r}+m_{1,1r}\right)$$
$$-\omega\dot{\delta}_r\left(m_{2,1r}-m_{1,2r}\right)-\omega\delta_r\dot{\delta}_t\left(m_{12,rt}-m_{21,rt}\right) \quad (22)$$

2.3. Potential Energy

The potential energy is expressed by the well known relation:

$$E_p=\frac{1}{2}\int_V\left(\sigma_{11}\varepsilon_{11}+2\sigma_{12}\varepsilon_{12}+\sigma_{22}\varepsilon_{22}\right)dV\ . \quad (23)$$

Finally the potential energy is expressed by the equation:

$$E_p=\frac{1}{2}\{\delta\}^T[k]\{\delta\}\ . \quad (24)$$

2.4. Work

The generalized concentrated forces q_i, $i=\overline{1,p}$, produce the work:

$$W^c=q_i\delta_i \quad ; \quad i=\overline{1,p}\ . \quad (25)$$

The generalized volume forces q_i^*, $i=\overline{1,p}$, produce the work:

$$W^{\mathrm{d}}=q_i^*\delta_i \quad ; \quad i=\overline{1,p}\ , \quad (26)$$

and the total work is:

$$W=\left(W^c+W^{\mathrm{d}}\right)=\left(q_i+q_i^*\right)\delta_i \quad ; \quad i=\overline{1,p}\ . \quad (27)$$

2.5 Energy of acceleration

Energy of acceleration is defined by:

$$E_a=\frac{1}{2}\int_V\rho a^2dV \quad . \quad (28)$$

In the local coordinate system we have:

$$\ddot{x}_{M'}=\ddot{x}_O+\left(x+N_{1j}\ddot{\delta}_j\right)-\omega^2\left(x+N_{1j}\delta_j\right)$$
$$-\varepsilon\left(y+N_{2j}\delta_j\right)-2\omega\left(y+N_{2j}\dot{\delta}_j\right). \quad (29)$$

$$\ddot{y}_{M'}=\ddot{y}_O+\left(y+N_{2j}\ddot{\delta}_j\right)-\omega^2\left(y+N_{2j}\delta_j\right)$$
$$+\varepsilon\left(x_1+N_{1j}\delta_j\right)+2\omega\left(x_1+N_{1j}\dot{\delta}_j\right). \quad (30)$$

Using Eq.(8) and (9) or (29) and (30) it is possible to obtain the expression of the energy of acceleration.

3. GIBBS-APPELL EQUATIONS

In the following will be presented shortly the Gibbs-Appell method. Let's consider a general a mechanical system composed by *N* material points. The positions of these points are defined by *n* independent coordinates, be these q_i, where $i=\overline{1,n}$. If the liaisons between these points are not dependent on time (scleronoma liaisons), the accelerations can be expressed as:

$$\bar{a}_i=\sum_k\sum_j\frac{\partial^2\bar{r}_i}{\partial q_k\partial q_j}\dot{q}_k\dot{q}_j+\sum_k\frac{\partial\bar{r}_i}{\partial q_k}\ddot{q}_k \quad ; \quad k,j=\overline{1,n} \quad ; \quad i=\overline{1,N} \quad (31)$$

where $\bar{r}_i$ represents the position vector of the point *i*.

The energy of acceleration denoted by *S* is defined as [48] and is:

$$S=\frac{1}{2}\sum_i m_i a_i^2 \quad ; \quad i=\overline{1,N}\ . \quad (32)$$

From formal point of view Eq.(32) is similar to that used to define the kinetic energy.

Eq.(32) can be extended too for an arbitrary point of the rigid as:

$$S=\frac{1}{2}\int_V a^2dm\ . \quad (33)$$

euivalent with Eq.(21). The acceleration of a current point of the rigid is:

$$\bar{a}=\sum_k\sum_j\frac{\partial^2\bar{r}}{\partial q_k\partial q_j}\dot{q}_k\dot{q}_j+\sum_k\frac{\partial\bar{r}}{\partial q_k}\ddot{q}_k \quad ; \quad k,j=\overline{1,n} \quad (34)$$

Introducing Eq.(34) into Eq.(33) we get:

$$S=\frac{1}{2}\int_V\rho a^2dV=\frac{1}{2}\int_V\rho\left(\sum_k\sum_j\frac{\partial^2\bar{r}}{\partial q_k\partial q_j}\dot{q}_k\dot{q}_j+\sum_k\frac{\partial\bar{r}}{\partial q_k}\ddot{q}_k\right)^2dV=$$
$$E_{ao}(\dot{q})+E_{a1}(\dot{q},\ddot{q})+E_{a2}(\ddot{q}) \quad , \quad k,j=\overline{1,n}\ . \quad (35)$$

The following notations were used:

$$E_{ao}(\dot{q})=\frac{1}{2}\sum_k\sum_j\sum_l\sum_m\left(\int_V\rho\frac{\partial^2\bar{r}}{\partial q_k\partial q_j}\frac{\partial^2\bar{r}}{\partial q_l\partial q_m}dV\right)\dot{q}_k\dot{q}_j\dot{q}_l\dot{q}_m \quad ; \quad j,k,l,m=\overline{1,n} \quad (36)$$

$$E_{a1}(\dot{q},\ddot{q})=\frac{1}{2}\sum_k\sum_j\sum_l\left(\int_V\rho\frac{\partial^2\bar{r}}{\partial q_k\partial q_j}\frac{\partial\bar{r}}{\partial q_l}dV\right)\dot{q}_k\dot{q}_j\ddot{q}_l \quad ; \quad k,j,l=\overline{1,n} \quad (37)$$

$$E_{a2}(\ddot{q})=\frac{1}{2}\sum_k\sum_j\left(\int_V\rho\frac{\partial\bar{r}}{\partial q_k}\frac{\partial\bar{r}}{\partial q_j}dV\right)\ddot{q}_k\ddot{q}_j \quad ; \quad k,j=\overline{1,n} \quad (38)$$

The classic forms of the GA equations are:

$$\frac{\partial S}{\partial\ddot{q}_i}=Q_i \quad i=\overline{1,n} \quad (39)$$

In Eq.(39), Q_i represents the generalized force corresponding to the generalized coordinate q_i.

These equations will be used to determine the equation of motion of a multibody system with elastic elements and the material having voids.

$$\frac{\partial E_a}{\partial\ddot{\delta}_r}=Q_r \quad r=\overline{1,p} \quad , \quad (40)$$

where $Q_r = (k_{rt} + k_{v,rt})\delta_t + q_r + q_r^*$. Developing Eq.(35), it is possible to identify the terms:

$$E_{a2} = \frac{1}{2}(m_{rt} + m_{v,rt})\ddot{\delta}_r\ddot{\delta}_t \quad r,t = \overline{1,p} \tag{41}$$

$$E_{a1} = \ddot{x}_{O,i}\ddot{\delta}_r m^I_{O,ir} + (\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ij}\ddot{\alpha}_{jk})\ddot{\delta}_r m_{k,mr} + (\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ij}\ddot{\alpha}_{jk})\delta_r\ddot{\delta}_t m_{kr,mt} + 2\alpha_{ji}\dot{\alpha}_{jk}\dot{\delta}_r\ddot{\delta}_t m_{kr,mt}dV \tag{42}$$

E_{a0} is the term that does not contain generalized accelerations and play no role in obtaining the Gibbs–Appell equations. So the energy of acceleration becomes:

$$E_a = E_{a0} + E_{a1} + E_{a2} \quad . \tag{43}$$

Equation (40) can be written as:

$$\frac{\partial(E_{a1} + E_{a2})}{\partial\ddot{\delta}_r} = Q_r \quad r = \overline{1,p} \quad . \tag{44}$$

The generalized force vector is:

$$Q_r = (k_{rt} + k_{v,rt})\delta_t + q_r + q_r^* \quad ; \quad r,t = \overline{1,p} \quad . \tag{45}$$

It results:

$$\frac{\partial E_{a2}}{\partial\ddot{\delta}_r} = (m_{rt} + m_{v,rt})\ddot{\delta}_t \quad r,t = \overline{1,p}\,; \tag{46}$$

and:

$$\frac{\partial E_{a1}}{\partial\ddot{\delta}_r} = 2\omega\dot{\delta}_r(m_{12,rt} - m_{21,rt}) + [-\varepsilon\delta_r(m_{12,rt} - m_{21,rt}) - \omega^2 m_{rt}]\delta_t + (\ddot{x}_{O,1}m^I_{O,1r} + \ddot{x}_{O,2}m^I_{O,2r}) - \varepsilon(m_{2,1r} - m_{1,2r}) + \omega(\dot{x}_{O,1}m^I_{O,2r} + \dot{x}_{O,2}m^I_{O,1r}) - \omega^2(m_{2,2r} + m_{1,1r}) \quad ; \quad r,t = \overline{1,p} \quad . \tag{47}$$

We have the relations:

$$\frac{\partial E_{a0}}{\partial\delta_r} = 0 \quad ; \quad r = \overline{1,p} \quad . \tag{48}$$

Finally, it results:

$$(m_{rt} + m_{v,rt})\ddot{\delta}_t + 2\omega\dot{\delta}_r(m_{12,rt} - m_{21,rt}) + [k_{rt} + k_{v,rt} - \varepsilon(m_{12,rt} - m_{21,rt}) - \omega^2\delta_t m_{rt}]\delta_t = -(\ddot{x}_O m^I_{O,1r} + \ddot{y}_O m^I_{O,2r}) + \varepsilon(m_{2,1r} - m_{1,2r}) - \omega(\dot{x}_{O,1}m^I_{O,2r} + \dot{x}_{O,2}m^I_{O,1r}) + \omega^2(m_{2,2r} + m_{1,1r}) - q_r - q_r^* = 0 \quad ; \quad r,t = \overline{1,p}\,. \tag{49}$$

4. DISCUSSION AND CONCLUSIONS

FEM represents a method widely used at the moment for the study of mechanical systems with elastic elements. It incorporates a large base of results and situations reported in the specialized literature by numerous researchers. The method lends itself very well to highly complex systems. Within this method, calculation procedures and modules have been developed that can accurately capture different aspects of the behavior of an elastic body in the case of different loading cases. In the case of elastic systems where the dynamic aspect can no longer be neglected, it is necessary to use a suitable model to obtain the equations of motion for a single finite element, equations then used in the analysis of the entire system. But obtaining these equations involves difficulties related to the complexity or the particular aspects of the studied application. In engineering practice, several equivalent formalisms are used that finally provide the equations of motion. The planar movement of an MBS represents a particular case in the analysis of such systems. The particularities presented by such a method allow obtaining these equations in a simpler form, an objective that was also pursued in the present work. The existence of voids in the materials used to manufacture the studied MBS implies the introduction of additional terms in the motion equations. Pursuing these objectives, the paper proposes an alternative method for obtaining these equations. Among the options offered by Analytical Mechanics for this, the Gibbs-Appell equations were chosen, for reasons also presented in the Introduction section. Alternative methods as Lagrange's Equations, Maggi's equations or Kane equations are equivalent and ultimately provide the same set of equations of motion, but the particularities presented by some of these methods make them easier to use in different circumstances. The choice of one of the methods offered by Mechanics is generally determined by the experience of the researcher and his familiarity with the main concepts used. The use of Gibbs-Appell equations has the advantage of reducing the number of differential equations. This presents an economic advantage that, for complex systems such as those used in modern industry, with a large number of degrees of freedom, leads to the reduction of modeling and simulation costs. A difficulty for most researchers is the introduction of the notion of acceleration energy. It is a concept with which researchers are less familiar. In the use of Lagrange's equations, very common notions are used and with which researchers are very familiar, such as kinetic, potential or work energy.

Voids in materials are generally an undesirable phenomenon that occurs during manufacturing and leads to a decrease in mechanical properties. In the context of the strong development of the use of composite materials, the study of their effects has become a necessity. However, such a study is extremely difficult to perform, involving numerous calculation aspects that must be taken into consideration. For this reason, the literature that addresses this problem and that was presented in the

Introduction section is still poor in results. Any possibility to make this calculation easier and to use opportunities arising in the reconsideration of some classic modeling methods is welcome. In this framework, the reconsideration of the Gibbs-Appel equations for modeling and writing the motion equations of such materials represents a new element that facilitates the analysis. The number of operations involved in the modeling phase is reduced and also the number of arithmetic operations related to them is reduced three times. Obviously, the analysis time of a system will not be reduced by three times, but a significant decrease in the time for the study of such problems will be observed.

In conclusion, the advantages presented by this method consist in reducing the number of operations required for writing the equations of motion. If it is taken into account that in the finite element method one works with systems of very large dimensions, with many degrees of freedom, this facility offered by the Gibbs-Appell method can determine the reduction of the time required for modeling the problem to be solved and the reduction of computer time (operations which will of course lead to a decrease in the cost price in the case of the analysis of such problems).

REFERENCES

[1] Judd N.C.W. and Wright W.W. Voids and their effects on mechanical-properties of composites—appraisal. *Sampe J.* **1978**, *14*, pp. 10–14.

[2] Hernandez, S.; Sket, F.; Gonzalez, C.; Llorca, J. Optimization of curing cycle in carbon fiber-reinforced laminates: void distribution and mechanical properties. *Compos. Sci. Technol.* **2013**, *85*, pp. 73–82.

[3] Campbell, F.C.; Mallow, A.R.; Browning, C.E. Porosity in carbon-fiber composites—an overview of causes. *J. Adv. Mater.* **1995**; 26, pp. 18–33.

[4] Naganuma, T.; Naito, K.; Kyono, J.; Kagawa, Y. Influence of prepreg conditions on the void occurrence and tensile properties of woven glass fiber-reinforced polyimide composites. *Compos. Sci. Technol.* **2009**, *69*, 2428–2433.

[5] Gladysz, G.M.; Chawla, K.K. *Intrinsic Voids in Polymers. Voids in Materials: From Unavoidable Defects to Designed Cellular Materials*; Elsevier: Amsterdam, The Netherlands, 2015; pp. 37–48.

[6] Sung, B.J.; Yethiraj, A. Structure of void space in polymer solutions. *Phys. Rev. E* **2010**, *81*, 237.

[9] Tang, J.-M.; Lee, W.I.; Springer, G.S. Effects of cure pressure on resin flow, voids, and mechanical properties. *J. Compos. Mater.* **1987**, *21*, pp. 421–440.

[10] Liu, L.; Zhang, B.-M.; Wang, D.-F.; Wu, Z.-J. Effects of cure cycles on void content and mechanical properties of composite laminates. *Compos. Struct.* **2006**, *73*, 303–309.

[11] Ghiorse, S. Effect of void content on the mechanical properties of carbon/epoxy laminates. *Sample Q.* **1993**, *24*, pp. 54–59.

[12] Goodman, M.A.; Cowin, S.C. A continuum theory for granular materials. *Arch. Ration. Mech. Anal*. **1972**, *44*, pp. 249–266.

[13] Chirita, S.; Arusoaie, A. Thermoelastic waves in double porosity materials. Eur. J. Mech. A Solids 2021, 86, 104177.

[14] Modrea, A.; Vlase, S.; Calin, M.R.; Peterlicean, A. The influence of dimensional and structural shifts of the elastic constant values in cylinder fiber composites. *J. Optoelectron. Adv. Mater*. **2013**, *15*, 278–283.

[15] Vlase, S; Purcarea, R; Mihalcica, M. Behavior of a new **H**eliopol/Stratimat 300 composite laminate**.** Optoelectronics and Advanced Materials-Rapid Communications, 2013, 7 (7-8) , pp.569-572.

[16] Öchsner, A. *Computational Statics and Dynamics: An Introduction Based on the Finite Element Method*; Springer: Singapore, 2020.

[17] Iesan, D. *Thermoelastic Models of Continua*; Kluwer Academic: Dordrecht, The Netherlands, 2004.

[18] Nguyen, V.T.; Hwu, C. A Review on Modelling Polymer Composites via Anisotropic Viscoelasticity. *J. Aeronaut. Astronaut. Aviat.* **2021**, *53*, 387–402. https://doi.org/10.6125/JoAAA.202109_53.05

[19] Raju, K.; Tay, T.E.; Tan, V.B.C. A review of the FEM method for composites. *Multiscale Multidiscip. Model. Exp. Des.* **2021**, *4*, 1–24. https://doi.org/10.1007/s41939-020-00087-x

[20] Sharma, A.; Bagha, A.K.; Shukla, D.K.; Bahl, S. Finite element model updating of metallic and composite structures-A state of the art review. *AIMS Mater. Sci.* **2021**, *8*, 390–415. https://doi.org/10.3934/ matersci.2021025

[21] Muzel, S.D.; Bonhin, E.; Guimaraes, N.M.; Guidi, E.S. Application of the Finite Element Method in the Analysis of Composite Materials: A Review. *Polymers* **2020**, *12*, 818. https://doi.org/10.3390/ polym12040818

[22] Vlase, S.; Negrean, I.; Marin, M.; Năstac, S. Kane's Method-Based Simulation and Modeling Robots with Elastic Elements, Using Finite Element Method. *Mathematics* **2020**, *8*, 805. https://doi.org/ 10.3390/math8050805

[23] Vlase,S., Teodorescu,P.P.,Itu, C., Scutaru, M.L. Elasto-dynamics of a solid with a general "rigid" motion – II. Rom. Journ. Phys., Vol. 58, Nos. 7–8, P. 882–892, Bucharest, 2013.

[24] Niculiță, C. et al Optimum stacking in a multi-ply laminate used for the skin of adaptive wings. *Optoelectron. Adv. Mater. Rapid Commun*. **2011**, *5*, 1233–1236.

[25] Nunziato, J.W.; Cowin, S.C. Linear elastic materials with voids. *J. Elast.* **1983**, *13*, 125–147.

[26] Chang, X.; Ren, M.F.; Li, T.; Guo, X. Evaluation of mechanical behaviour of unidirectional fibre-reinforced composites considering the void morphology. *J. Reinf. Platics Compos.* **2017**, *36*, 1817–1828. https://doi.org/ 10.1177/ 0731684417727358.

[27] Dong, C.S. Effects of Process-Induced Voids on the Properties of Fibre Reinforced Composites. *J. Mater. Sci. Technol.* **2016**, *32*, 597–604. https://doi.org/10.1016/ j.jmst.2016.04.011.

[28] Erdman, A.G.; Sandor, G.N.; Oakberg, A. A General Method for Kineto-Elastodynamic Analysis and Synthesis of Mechanism. Journal of Engineering for Industry. *ASME Trans*. **1972**, *94*, 1193.

[29] Bagci, C. Elastodynamic Response of Mechanical Systems using Matrix Exponential Mode Uncoupling and Incremental Forcing Techniques with Finite Element Method. In Proceedings of the Sixth Word Congress on Theory of Machines and Mechanisms, New Delhi, India, 15–20 December 1983; p. 472.

[30] Bahgat, B.M.; Willmert, K.D. Finite Element Vibrational Analysis of Planar Mechanisms. *Mech. Mach. Theory* **1976**, *11*, 47.

[31] Cleghorn, W.L.; Fenton, E.G.; Tabarrok, K.B. Finite Element Analysis of High-Speed Flexible Mechanisms. *Mech. Mach. Theory* **1981**, *16*, 407.

[32] Midha, A.; Erdman, A.G.; Frohrib, D.A. Finite element approach to mathematical modeling of high-speed elastic linkages. *Mech. Mach. Theory* **1978**, *13*, pp. 603–618.
[33] Nath, K.; Ghosh, A. Steady-state response of mechanism with elastic links by finite element methods. *Mech. Mach. Theory* **1980**, *15*, 199.
[34] Marin, M., Chirila, A., Öchsner, A. *et al.* About finite energy solutions in thermoelasticity of micropolar bodies with voids. *Bound Value Probl* **2019**, 89 (2019). https://doi.org/10.1186/s13661-019-1203-3
[35] Vlase, S. Dynamical Response of a Multibody System with Flexible Elements with a General Three-Dimensional Motion. *Rom. J. Phys.* **2012**, *57*, pp. 676–693
[36] Vlase, S.; Negrean, I.; Marin, M.; Năstac, S. Kane's Method-Based Simulation and Modeling Robots with Elastic Elements, Using Finite Element Method. *Mathematics* **2020**, *8*, 805. https://doi.org/ 10.3390/math8050805
[37] Hou, W.; Zhang, X. Dynamic analysis of flexible linkage mechanisms under uniform temperature change. *J. Sound. Vib.* **2009**, *319*, pp. 570–592.
[38] Neto, M.A.; Ambrósio, J.A.C.; Leal, R. Composite materials in flexible multibody systems. *Comput. Methods Appl. Mech. Eng.* **2006**, *195*, pp. 6860–6873.
[39] Piras, G.; Cleghorn, W.L.; Mills, J.K. Dynamic finite-element analysis of a planar high-speed, high-precision parallel manipulator with flexible links. *Mech. Mach. Theory* **2005**, *40*, pp. 849–862.
[40] Gibbs, J.W. On the fundamental formulae of dynamics. *Am. J. Math.* **1879**, V.2, pp. 49–64.
[41] Appell, P. Sur une forme générale des equations de la dynamique. *C.R. Acad. Sci. Paris.* **1899**, *1900*, 129.
[42] Mirtaheri, S.M.; Zohoor, H. The Explicit Gibbs-Appell Equations of Motion for Rigid-Body Constrained Mechanical System. In Proceedings of the 2018 6th RSI international conference on robotics and mechatronics (IcRoM), Tehran, Iran, 23–25 October 2018; pp. 304–309.
[43] Korayem, M.H.; Dehkordi, S.F. Motion equations of cooperative multi flexible mobile manipulator via recursive Gibbs-Appell formulation. *Appl. Math. Model.* **2019**, *65*, pp. 443–463.
[44] Amini, S.; Dehkordi, S.F.; Fahraji, S.H. Motion equation derivation and tip-over evaluations for K mobile manipulators with the consideration of motors mass by the use of Gibbs-Appell formulation. In Proceedings of the 5th RSI International Conference on Robotics and Mechatronics (IcRoM), Tehran, Iran, 25–27 October 2017.
[45] Korayem, M.H.; Dehkordi, S.F. Derivation of dynamic equation of viscoelastic manipulator with revolute-prismatic joint using recursive Gibbs-Appell formulation. *Nonlinear Dyn.* **2017**, 89, pp. 2041–2064.
[46] Shafei, A.M.; Korayem, M.H. Theoretical and experimental study of dynamic load-carrying capacity for flexible robotic arms in point-to-point motion. *Optim. Control. Appl. Methods* **2017**, 38, pp. 963–972.
[47] Vlase, S.; Negrean, I.; Marin, M.; Scutaru, M.L. Energy of Accelerations Used to Obtain the Motion Equations of a Three- Dimensional Finite Element. *Symmetry* **2020**, *12*, 321.
[48] Negrean, I.; Crișan, A.-D.; Vlase, S. A New Approach in Analytical Dynamics of Mechanical Systems. *Symmetry* **2020**, *12*, 95.
[49] Vlase, S. Elimination of Lagrangian Multipliers. *Mech. Res. Commun.* **1987**, *14*, 17.
[50] Scutaru, M.L.; Mitrica, B. Dynamical Analysis of the Mechanical System with Two Degrees of Freedom Applied to the Transmission of the Wind Turbine. *Math. Probl. Eng.* **2016**, *2016*, 3821083
[51] Katouzian, M. et al Finite Element Method-Based Simulation Creep Behavior of Viscoelastic Carbon-Fiber Composite. *Polymers* **2021**,13 (7)
[52] Codarcea-Munteanu, L; Marin, M and Vlase, S. The study of vibrations in the context of porous micropolar media thermoelasticity and the absence of energy dissipation. *Journal of Computational Applied Mechanics* 2023, 54 (3), pp.437-454
[53] Scutaru, M.L.; Vlase, S. and Marin, M. Symmetrical Mechanical System Properties-Based Forced Vibration Analysis. *Journal of Computational Applied Mechanics*, 54 (4), pp.501-514, 2023.
[54] Bratu, P, Nitu, CM and Tonciu, O. Effect of Vibration Transmission in the Case of the Vibratory Roller Compactor. *Romanian Journal pf Acoustic and Vibration*, 20 (1), pp.67-72
[55] Vlase, S. et al Vibration analysis of the Gamma-Ray element in the ELI-NP interaction chamber (IC). *Journal of Computational Applied Mechanics* 2024, 55 (2) , pp.275-288.
[56] Marin, M., Öchsner, A., Vlase, S., Grigorescu, D.O., Tuns, I.: Some results on eigenvalue problems in the theory of piezoelectric porous dipolar bodies, *Contin. Mech. Thermodyn*. 35, pp. 1969–1979 (2023), https://doi.org/ 10.1007/ s00161-023-01220-0
[57] Marin, M., Öchsner, A., Bhatti, M.M., Some results in Moore-Gibson-Thompson thermoelasticity of dipolar bodies, *ZAMM-Journal of Applied Mathematics and Mechanics*, 100(12), Art No. e202000090, 2020
[58] Marin, M., Hobiny, A., Abbas, I. Finite element analysis of nonlinear bioheat model in skin tissue due to external thermal sources, *Mathematics*, 9(13), Art. No. 1459, 2021
[59] Itu, C.; Scutaru, M.L.; Vlase, S. Elastic Constants of Polymeric Fiber Composite Estimation Using Finite Element Method. *Polymers* **2024**, *16*, 354. https://doi.org/10.3390/polym 16030354
[60] Vlase, S., Marin, M. Finite Element Method-Based Dynamic Response of Micropolar Polymers with Voids. Polymers, 13(21), 2021, doi 10.3390/polym13213727.