
Vibration Of a Beam on a Variable Winkler Elastic Foundation Using The Spectral Element Method

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Abstract: - This study investigates the free vibration problem of Euler-Bernoulli beams resting on variable Winkler elastic foundations using the spectral element method. The soil stiffness under the foundation is assumed to vary as constant, linear, or parabolic along the beam's longitudinal axis. Based on Euler-Bernoulli beam theory, the governing partial differential equation is derived and solved in the frequency domain. The spatial solution is obtained by solving the fourth-order linear homogeneous differential equation, while an integral formulation is employed to determine the particular solution. The obtained solution is then used to examine the effects of the variable elastic foundation, beam-soil interaction, and the intensity of variation of the Winkler foundation on the nonlinear natural frequencies of a cantilevered beam. Finally, the computed results are compared with available reference data, demonstrating the accuracy and robustness of the proposed numerical approach.

Keywords: - Free vibrations, Spectral element method, Variable Winkler elastic foundation, Finite element method, Soil-structure interaction, Properties of soil foundation.

1. INTRODUCTION

Beams resting on elastic media have numerous applications across various engineering disciplines, including transportation, marine, pavement, aerospace, mechanical, petroleum, geotechnical, and civil engineering [1]. Modeling the interfacial continuum between the beam and the supporting soil foundation is a crucial aspect for analyzing such problems [2]. The modeling process becomes more complex when the structure is influenced by its surrounding environment. In general, the dynamic analysis of beam-soil interaction is challenging due to the inherent complexity of soil behavior, the interface continuum, structural characteristics, and the dynamic load process.

The surrounding soil increases the resistance of buried structures such as pipelines and significantly influences their dynamic response. Many engineering problems can be modeled using beam-on-elastic-foundation theories, such as the Winkler and Pasternak models. In the Winkler model, the vertical displacement is assumed to be proportional to the contact pressure at each point [3], and the soil

foundation is modeled by a series of closely spaced, mutually independent linear elastic springs.

Under static loadings, deep beams resting on foundations exhibiting both linear and nonlinear behavior were modeled using the ANSYS package [4]. In that study, nonlinear springs were employed to represent the nonlinear behavior of the soil, and the result showed that the maximum deflection increased by 2.5% when the foundation was modeled as a beam on a nonlinear Winkler model compared to a linear Winkler foundation. In the present work, an analytical solution of the partial differential equation of a simply supported Euler-Bernoulli beam on a variable Winkler elastic foundation is developed. The nonlinearity is characterized by a negative fourth-power variation of the stiffness parameter [5].

Numerous studies have addressed the dynamic analysis of beams resting on a linear Winkler elastic foundation, and this topic is extensively documented in the literature. Engineering problems of this type are typically governed by partial differential equations, most of which do not admit closed-form analytical solutions. Consequently, numerical

methods are often employed to obtain approximate solutions. The behavior of beams on elastic foundations has been investigated using various analytical and numerical approaches to predict overall structural responses. In this context, Doyle et al. [6] analyzed the free vibration of beams partially supported by an elastic foundation, accounting for the bending stiffness of the beam. Furthermore, the differential transform method (DTM) has been employed to compute the non-dimensional natural frequencies of tapered cantilever Bernoulli–Euler beams [7-8].

Analytical solutions of bending beams on elastic foundations have been obtained using singularity functions, an approach that combines computational algebra systems with efficient beam analysis techniques [9]. Furthermore, Sato et al. [10] developed a mathematical model to investigate the static and dynamic responses of beams on equidistant elastic supports (BOES). This formulation enables a smooth transition from BOES conditions to those of a beam on an elastic foundation (BOEF). However, these approaches are generally limited to specific ranges of support stiffness, spacing, and flexural rigidity of the beam.

In the numerical domain, the transverse harmonic excitation coupled with the axial motion of a cracked beam has been investigated using a structure-preserving method. The effect of cracking on the beam's flexibility was quantified through the concept of equivalent bending stiffness [11]. Furthermore, studies on the active control of dynamic responses in spatial structures have addressed the damping of spatial vibrations and wave propagation phenomena. A model based on the vibration and wave propagation characteristics of a spatial flexible damping panel attached to four special springs was proposed and analyzed [12]. In this case, by considering the coupling among structural vibration, attitude dynamics, and orbital dynamics, a comprehensive dynamic model of the flexible damping panel-spring-mass system was established. Wang et al. [13] developed the weak-form quadrature element method (QEM), which combines the finite element method with the accuracy of spectral techniques. Their work provided a comprehensive review of QEM applications to various problems and engineering problems. Additionally, the coupling effects between planar motion and flexible vibration of a free plate were also investigated using advanced numerical formulations. The mathematical model describing the dynamic problem of a spatial plate was formulated using the single-variable simple thick plate theory [14].

The Natural frequencies of non-uniform beams resting on an elastic foundation were computed using the spline collection procedure [15]. This method employs the differential quadrature technique to discretize the governing differential equation of each element and to enforce the transition conditions at inter-element boundaries [16]. The natural frequencies and mode shapes of Euler-Bernoulli beams on a simple Winkler linear elastic foundation were also determined using He's variational iteration method, developed for various boundary conditions [17].

Moreover, Zhou [18] presents a general vibratory solution for beams on a variable Winkler elastic foundation. In this approach, the reaction force of the foundation is treated as an external force acting on the beam. The analytical solution involves an integral expression, which is evaluated numerically using the trapezoidal rule. A similar objective was achieved using the differential transform method (DTM) based on the Taylor series expansion [19]; in addition, the influence of foundation parameters on dynamic response was examined using the finite element method (FEM).

Recently, the differential quadrature method has been employed to analyze the nonlocal vibration behavior of nano-beam resting on various types of Winkler elastic foundations, including constant, linear, parabolic, and sinusoidal variation [20]. In addition, the interaction problem of raft-pile-soil-structure has been studied using a combined boundary element method (BEM) and finite element method (FEM in the frequency domain, and the soil nonlinearity was modeled using the hyperbolic stress-strain relationship and the modified Drucker-Prager model [21]. Furthermore, recent investigations on nuclear power plants founded on soft layered soil have demonstrated a significant influence of soil-structure interaction on the seismic response. In this context, simplified linear spring constants were proposed to simulate the behavior of pile group foundations [22].

The originality of the present work lies in the development and implementation of a dedicated spectral element formulation that explicitly accounts for spatially varying foundation stiffness within a unified framework. Unlike conventional approaches, the proposed SEM formulation enables high-accuracy solutions using a minimal number of elements, significantly improving computational efficiency compared to the standard finite element approach. Furthermore, the study provides new parametric insights into the influence of stiffness variation intensity and distribution type (constant, linear, and parabolic) on modal characteristics,

highlighting mode-dependent sensitivity effects that are not fully addressed in previous works.

2. MATHEMATICAL FORMULATION

2.1. Background of the SEM

Many authors [23-27] have contributed to the development and advancement of the SEM. This numerical technique has been widely applied to the dynamic responses of structures, providing higher accuracy and computational efficiency compared to conventional methods such as the finite element method and boundary element method.

The concept of SEM was first introduced by Narayanan and Beskos [28], who derived the dynamic stiffness matrix of a beam element and obtained the corresponding solution using the fast Fourier transform (FFT). Subsequently, Fournier [29] applied SEM to model axisymmetric flows in the frequency domain. Owing to its high accuracy and efficiency, the spectral method has also been widely employed for structural damage detection and identification [30]. In addition, it has been used to simulate room acoustics through a wave-based numerical formulation built upon the spectral element method [31]. Furthermore, SEM has been applied to investigate the structural response of plane frames considering the soil-structure interaction [27].

2.2 Basis of the formulation

In the spectral element method, the governing partial differential equation of motion is transformed from the time domain to the frequency domain using the discrete Fourier transform (DFT). This transformation eliminates time as an explicit variable and introduces frequency as the primary mathematical parameter of the problem. Based on previous studies [32-35], it has been demonstrated that a single spectral finite element is often sufficient to obtain an accurate dynamic response for such problems. This efficiency significantly reduces the number of elements, degrees of freedom, computational time, and memory storage requirements compared to conventional finite element formulations [23]. One of the objectives of the present work is to apply the spectral element method to quantify the effect of soil-structure interaction on the free vibration behavior of beams resting on variable Winkler elastic foundations.

2.3 Description of the problem

A beam resting on a variable Winkler elastic foundation is shown in Figure 1. The corresponding

partial differential equation of a beam with a uniform cross-section and constant flexural rigidity, resting on variable Winkler elastic foundation is

$$c^2 \frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{K(x)}{\rho\Omega} v(x,t) = \frac{q(x,t)}{\rho\Omega} \quad 0 \leq x \leq L \quad (1)$$

where $c^2 = \frac{EI}{\rho\Omega}$ (EI is the flexural rigidity, $\rho\Omega$ is the mass per unit length and Ω is the cross-sectional area), L is the length, $K(x)$ is the variable Winkler elastic foundation, $v(x,t)$ is the transverse deflection at time t , and $q(x,t)$ is the external loading.

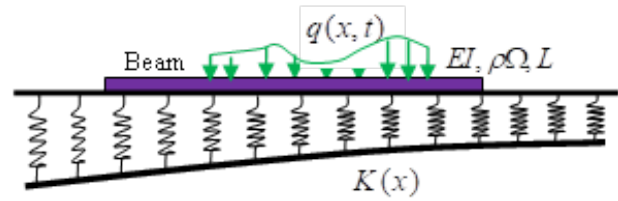


Figure 1. Beam resting on variable Winkler foundation

The soil under the foundation is modeled as a set of linear elastic springs with variable distribution along the longitudinal axis of the beam. The variation of Winkler elastic stiffness is considered using constant, linear, and parabolic distributions (Fig. 2).

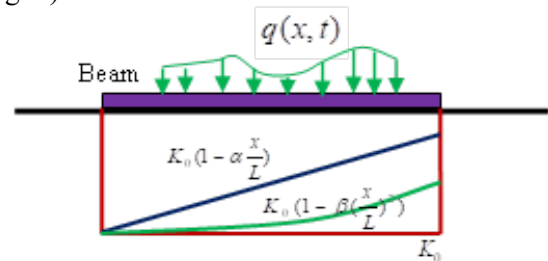


Figure 2. Elastic Winkler foundation modeling, $\alpha < \beta$

$$K(x) = K_0 \quad (2.1)$$

$$K(x) = K_0 \left(1 - \alpha \frac{x}{L}\right) \quad 0 \leq \alpha \leq 1 \quad (2.2)$$

$$K(x) = K_0 \left(1 - \beta \left(\frac{x}{L}\right)^2\right) \quad 0 \leq \beta \leq 1 \quad (2.3)$$

2.4 Numerical method

For the free vibrations of beams, the external forces are neglected and the equation (1) becomes

$$c^2 \frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{K(x)}{\rho\Omega} v(x,t) = 0 \quad (3)$$

Using the separation method of variables, the solution of equation (3) can be written as the product of spatial and temporal components.

$$v(x, t) = X(x) e^{i\omega t} \quad (4)$$

The spatial solution of equation (3) can be deduced as

$$\frac{d^4 X(x)}{dx^4} - \lambda^2 X(x) = -K(x)X(x) \quad (5)$$

$\lambda^4 = \frac{\rho\Omega}{EI} \omega^2 L^4$ called the non-dimensional natural frequency.

2.4.1 Spatial homogeneous solution

The homogeneous solution of differential equation (5) can be expressed as

$$X_h(x, \omega) = \langle \cos \lambda x \sin \lambda x \cosh \lambda x \sinh \lambda x \rangle \{a\} \quad (6)$$

$\{a\}$ are components of the homogeneous solution.

2.4.2 Spatial particular solution

The particular solution of the fourth-order differential equation (5) can be written [36] as,

$$X_p(x, \omega) = \frac{1}{2\lambda^3} \int_0^x K(s) X_h(s) [\sin \lambda(x-s) - \sinh \lambda(x-s)] ds \quad (7)$$

s is the length distribution of force on the beam.

Then, the combined solution can be expressed as

$$X(x, \omega) = \langle \cos \lambda x \sin \lambda x \cosh \lambda x \sinh \lambda x \rangle + \frac{1}{2\lambda^3} \int_0^x K(s) X_h(s) [\sin \lambda(x-s) - \sinh \lambda(x-s)] ds \{a\} \quad (8)$$

The degrees of freedom of the beam element are

$$v_1 = X(0, \omega), \theta_1 = \frac{\partial X(0, \omega)}{\partial x}, \quad v_2 = X(L, \omega) \text{ and } \theta_2 = \frac{\partial X(L, \omega)}{\partial x}.$$

Using equation 8, the generalized coefficients vector can be written

$$\{a\} = [B(\omega)]^{-1} \{q_e\} \quad (9)$$

The matrix $[B(\omega)]$ links the degrees of freedom vector $\{q_e\} = \langle v_1 \theta_1 v_2 \theta_2 \rangle^t$ of the beam to generalized variables $\{a\}$.

Expressions of shear force and bending moment at the ends of the beam can be written as, at $x=0$

$$T_1 = EI \frac{d^3 X(0, \omega)}{dx^3} \quad (10a)$$

$$M_1 = EI \frac{d^2 X(0, \omega)}{dx^2} \quad (10b)$$

at $x=L$

$$T_2 = -EI \frac{d^3 X(L, \omega)}{dx^3} \quad (10c)$$

$$M_2 = -EI \frac{d^2 X(L, \omega)}{dx^2} \quad (10d)$$

In matrix form, the equation (10) can be rewritten as

$$\{F_e\} = \begin{Bmatrix} T_1 \\ M_1 \\ T_2 \\ M_2 \end{Bmatrix} = [F(\omega)] \{a\} \quad (11)$$

Substituting equation (9) into equation (11), we obtain

$$\{F_e\} = [F(\omega)] [B(\omega)]^{-1} \{q_e\} \quad (12)$$

Parameters of the spectral stiffness matrix $[K_e(\omega)] = [F(\omega)] [B(\omega)]^{-1}$ can be computed numerically.

3. NUMERICAL EXAMPLES

Two computer programs, based on the SEM and the FEM were developed using MATLAB software to compute the natural frequencies of a beam resting on a variable Winkler elastic foundation. The proposed spectral formulation enhances the conventional FEM, providing high accuracy with a significantly lower computational cost. Several case studies have been conducted to investigate the influence of specially varying soil stiffness on the vibrational behavior of the beam, as described below.

3.1 Cantilevered beam on classical boundaries (without soil foundation)

In this study, a cantilever beam, both with and without Winkler elastic foundation, is investigated. This beam was previously analyzed by Kacar [19] using the differential transform method (DTM) and earlier by Rao [37] through an analytical solution. The geometrical dimensions and mechanical properties are illustrated in Figure 3.

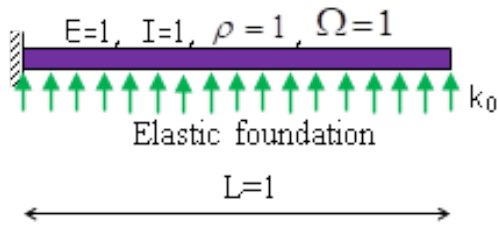


Figure 3. Cantilever beam resting with and without elastic foundation.

To demonstrate the accuracy of the proposed procedure, the free vibrations of an Euler-Bernoulli cantilever beam without a soil foundation are evaluated and compared with available results. In this case, the Winkler elastic foundation parameter is set to zero, and natural frequencies of the cantilever beam are computed. The results obtained with FEM and SEM are regrouped in Table 1 for the first four natural frequencies. As observed, the spectral element method provides satisfactory accuracy using a single element compared to the differential transform method [19, 37].

Table 1. First three natural frequencies: Comparison with DTM and analytical solution

Mode	$\omega_1 (rad / s)$	$\omega_2 (rad / s)$	$\omega_3 (rad / s)$
1 FE	3.532732	34.806893	-
50 FEs	3.516016	22.034493	61.697240
100 FEs	3.516016	22.034492	61.697216
150 FEs	3.516015	22.034492	61.697215
1SEM	3.516015	22.034491	61.697214
DTM [19]	3.516015	22.034910	61.697214
Analytical solution [37]	3.516005	22.034490	61.697207

3.2 Cantilever beam on constant Winkler elastic foundation

Then, the cantilever beam is assumed to be resting on a constant Winkler elastic foundation having a constant intensity K_0 (Fig. 4).

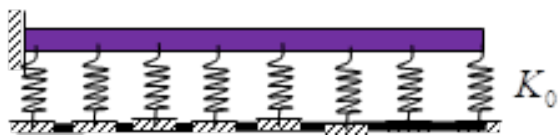


Figure 4. Cantilever beam on constant elastic foundation

The accuracy of SEM is validated by comparing the first three natural frequencies obtained by the finite element method and the differential transform method [19, 38] (Table 2).

Table 2. First three natural frequencies for beam resting on a constant elastic foundation

Mode	$\omega_1 (rad / s)$	$\omega_2 (rad / s)$	$\omega_3 (rad / s)$
1 FE	3.673979	34.835354	-
100 FEs	3.655113	22.057210	61.705323
150 FEs	3.655108	22.057212	61.705322
1SE	3.6552	22.0571	61.7051
DTM [19]	3.655438	22.057206	61.705266
Balkaya [38]	3.6554	22.0572	61.7053

The variation of natural frequencies versus elastic foundation stiffness K_0 is shown in figure 5. A partial influence of the Winkler elastic foundation stiffness on the natural frequencies is observed. This effect is particularly noticeable for $k_0 \leq 500$ in the first vibration mode and it widens to $k_0 \leq 1000$ for the second vibration mode. Beyond these intervals, the beam vibrates with constant natural frequencies, indicating that further increases in foundation stiffness have little effect on the dynamic response.

In conclusion, the natural frequencies are more affected by the mechanical properties of soil and this effect becomes more pronounced for $n \geq 3$ (n is the number of modes of vibration). For the first vibration mode, the dynamic response of a beam resting on soil ($K_0=1500$) is approximately four times higher than of $K_0=1$.

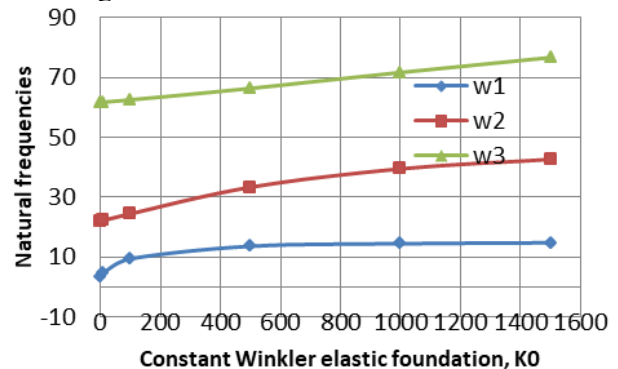


Figure 5. Natural frequencies of cantilever beam on constant Winkler elastic foundation

3.3 Cantilever beam on linear Winkler elastic spring

In this case, a linear Winkler elastic foundation is assumed with an intensity of variation (Fig. 6).

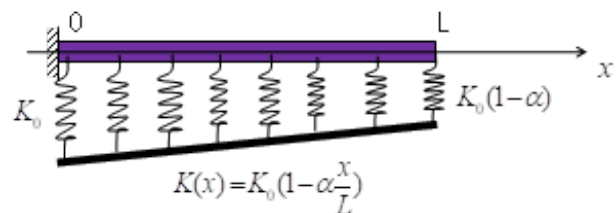


Figure 6. Cantilever beam on linear Winkler elastic foundation

For $\alpha = 0.20$, natural frequencies for the first three modes of vibration using SEM and FEM are regrouped in Table 3.

Table 3. First three natural frequencies for beam resting on a linear elastic foundation

Mode	$\omega_1 (rad / s)$	$\omega_2 (rad / s)$	$\omega_3 (rad / s)$
1 FE	3.745146	34.850163	-
100 FEs	3.693004	22.063580	61.707594
150 FEs	3.693003	22.063578	61.70760
1SE	3.693004	22.063580	61.707592
DTM [19]	3.633331	22.054482	61.704481

Due to the lack of available experimental results in the literature, the results obtained using the present approach were compared with those from the FEM and DTM [19]. As shown in Table 3, a close convergence among the three methods is observed, confirming the accuracy and reliability of the proposed spectral element formulations.

The influence of the linearly varying Winkler elastic foundation on the natural frequencies is then studied by considering two values of the variation intensity.

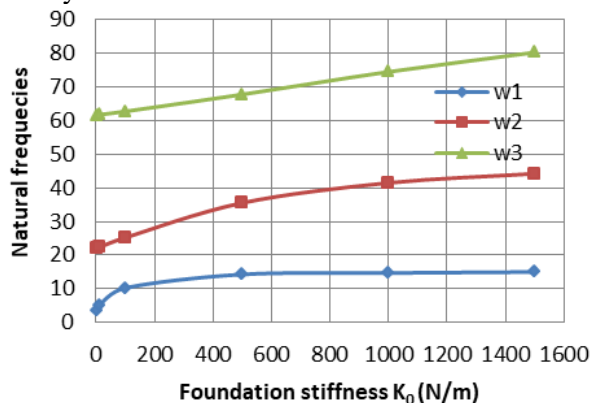


Figure 7. Natural frequencies versus linear Winkler elastic foundation with $\alpha = 0.20$

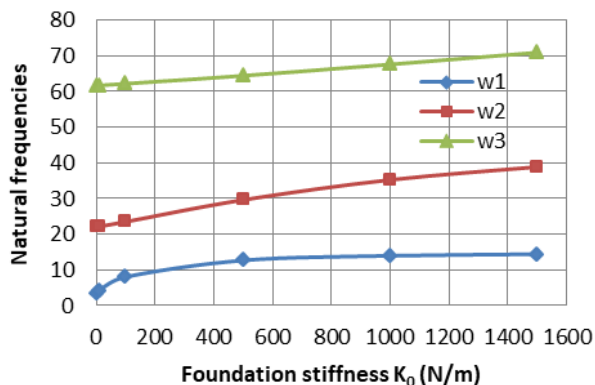


Figure 8. Natural frequencies for linear Winkler elastic foundation with $\alpha = 0.80$

In both cases, the natural frequencies depend on the linear Winkler elastic foundation, and dynamic attenuation depends on the degree of variation intensity. When the variation intensity increases, the dynamic attenuation of the beam decreases and becomes more pronounced from the first mode to the third mode of vibration (Figs. 7-8). For $k_0=1500$ N/m, $\omega_3 = 80.315 rad / s$ ($\alpha = 0.20$), the natural frequency decreases to $70.93 rad / s$ ($\alpha = 0.80$), representing an increase of 13%.

3.4 Cantilevered beam on parabolic Winkler elastic spring

In this case, a parabolic Winkler elastic foundation is considered in order to calibrate the influence of $K(x)$ -stiffness and the intensity of variation β (Fig. 9).

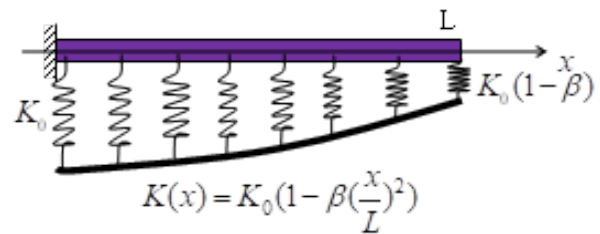


Figure 9. Cantilever beam on parabolic Winkler elastic foundation

Table 4. First three natural frequencies for the beam used

Mode	$\omega_1 (rad / s)$	$\omega_2 (rad / s)$	$\omega_3 (rad / s)$
1 FE	3.711801	34.843184	-
50 FEs	3.692331	22.063467	61.707577
100 FEs	3.692330	22.063466	61.707553
150 FEs	3.692330	22.063466	61.707550
1SE	3.692330	22.063466	61.707551
DTM 19]	3.636954	22.055327	61.704795

Table 4 shows the numerical results obtained using FEM, SEM, and DTM for a cantilever beam resting on a parabolically distributed Winkler elastic foundation. The mechanical properties of the continuum under the foundation influence the dynamic response of the beam. It is observed that the natural frequencies increase in the presence of the nonlinear term, and this effect becomes more pronounced as the variation mode number increases (Figs 7-8 and 10-11).

The variation of the natural frequencies with respect to the degree of foundation stiffness variation is depicted in Figure 12. For weak soil

conditions, the natural frequencies are particularly sensitive to the spatial distribution of the elastic foundation along the beam's length. In this case, as the variation intensity increases, the beam's dynamic flexibility decreases. This effect becomes more pronounced for the higher modes of vibration (Fig. 12).

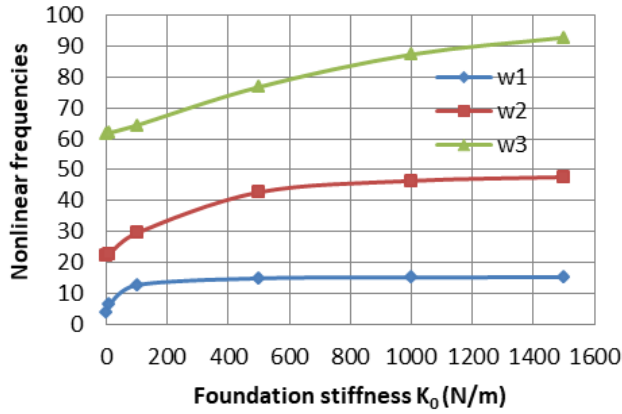


Figure 10. Natural frequencies versus parabolic Winkler elastic foundation with $\beta = 0.2$

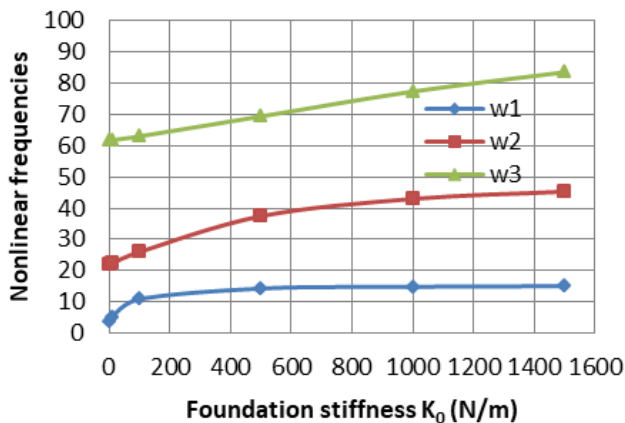


Figure 11. Nonlinear frequencies versus parabolic Winkler elastic foundation with $\beta = 0.80$

For beams resting on stiff soil ($k_0=1000$ N/m), the difference in natural frequencies between the two soil types is minimal in the first vibration mode for $\alpha = \beta$, however, it becomes more noticeable in the second and third modes. Moreover, the intensity of stiffness variation has a clear influence on the natural frequencies, and this effect becomes increasingly significant for the higher vibration modes (Fig. 13).

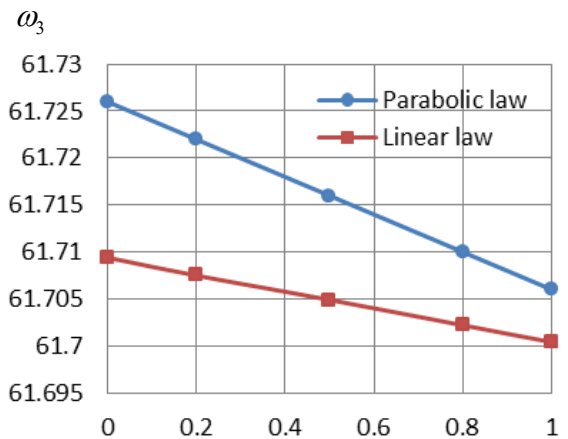
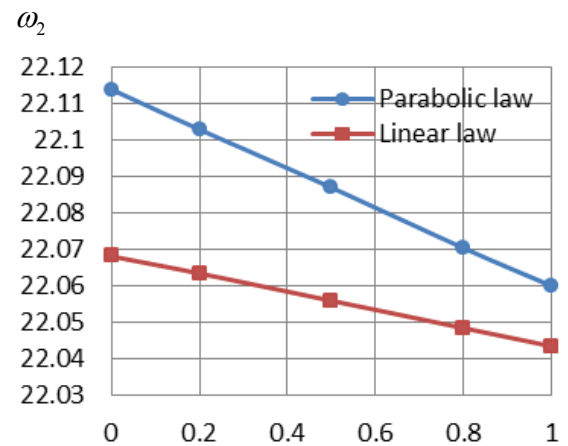
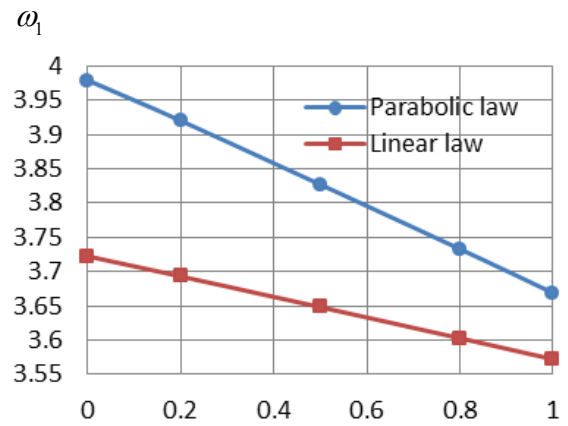


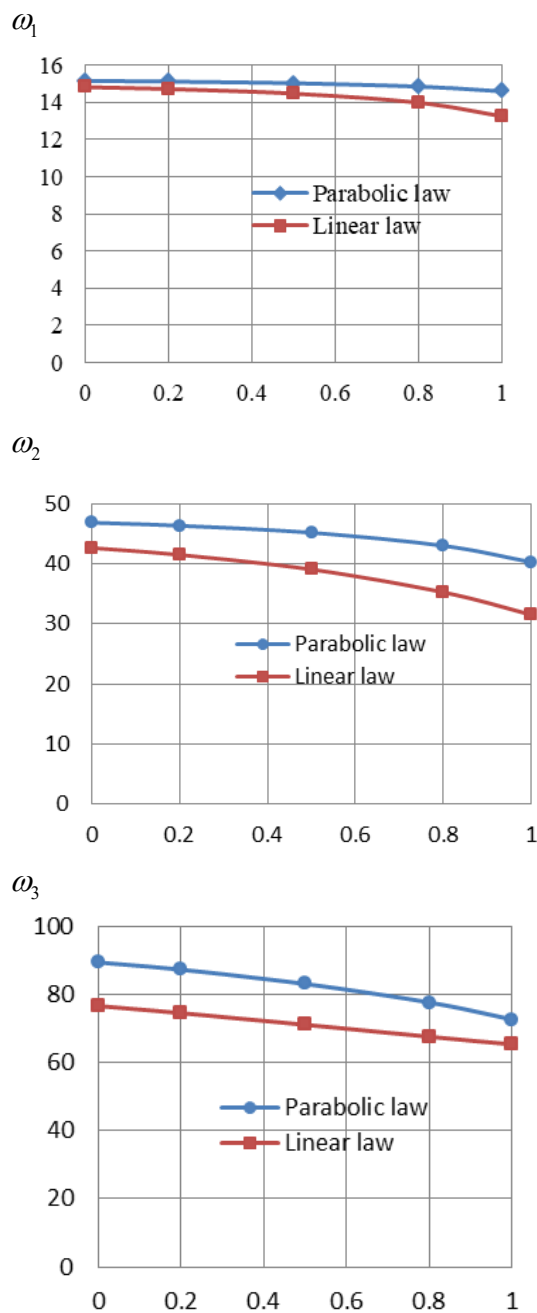
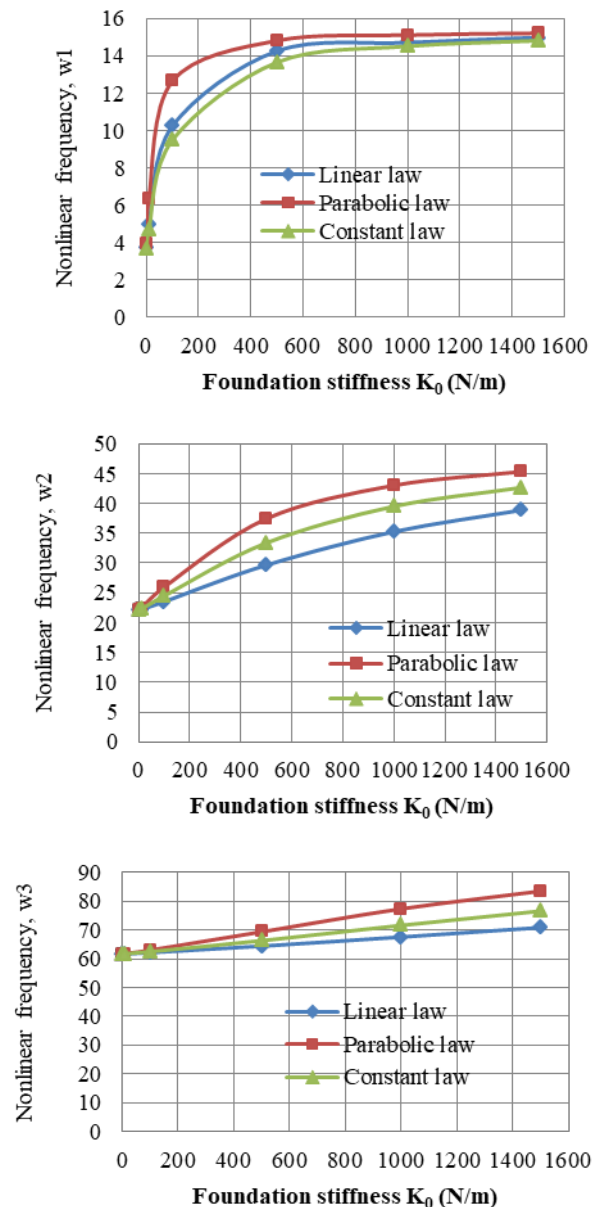
Figure 12. Natural frequencies versus variation intensity for $K_0=1$ N/m and $\alpha = \beta$

The first vibration mode is the most dominant in free vibrations and is significantly influenced by the soil properties. In this context, varying mechanical properties from $k_0=1$ N/m to $k_0=1000$ N/m, the natural frequency varies from 3.85 rad/s to 14.98 rad/s; it is 3.89% higher. Adding, as the intensity of variation varies from $\alpha = 0$ to 1, the frequency rate increases from 6.35% to 7.61%; it is 24.56% higher (Table 5).

Table 5. ω_1 versus k_0 and α

	ω_1 (rad / s)		
$\alpha = \beta$	0.0	1.0	Percentage
$K_0=1.0$	3.85	3.62	6.35%
$K_0=1000$	14.98	13.92	7.61%
Percentage	3.89%	3.84%	

Figure 14 shows the effect of K_0 on the nonlinear frequencies. Three distributions of the variable Winkler elastic foundation equation (2) are considered. As K_0 increases, the nonlinear frequency increases for all modes of vibration and for any value of K_0 .

**Figure 13.** Natural frequencies versus variation intensity for $K_0=1000$ N/m and $\alpha = \beta$ **Figure 14.** Effect of K_0 on nonlinear frequency for different distributions of elastic foundation

The parabolic distribution leads to higher natural frequencies compared to the linear and constant variations of the elastic foundation. For the second and third vibration modes, the beam resting on a constant elastic foundation exhibits higher natural frequencies than that on a linear Winkler elastic foundation for all values of K_0 . In contrast, the opposite trend is observed for the first vibration mode.

Finally, the free vibration mode shapes of the cantilever beam resting on a variable Winkler elastic foundation are found to be nearly identical, indicating that the beam exhibits similar behavior in both transverse and rotational vibrations. The normalized mode shapes of the first three modes of vibration are presented in Figure 15.

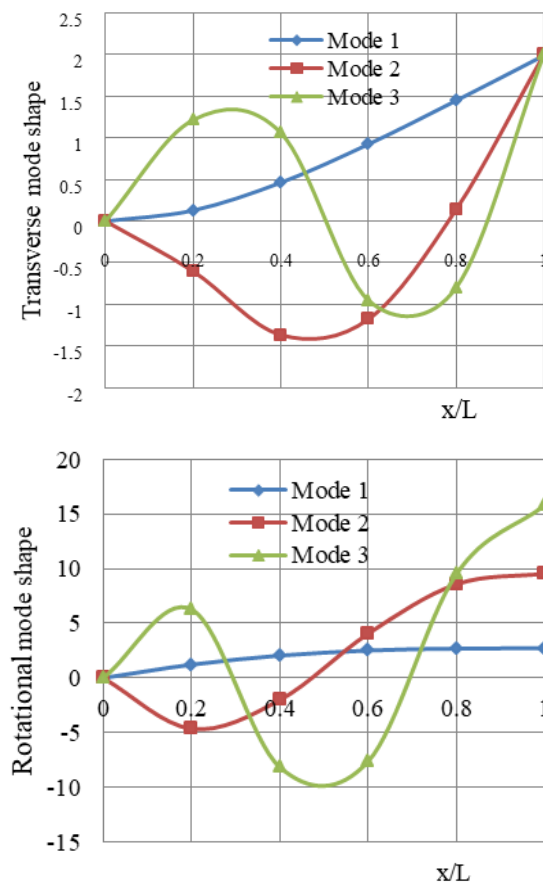


Figure 15. Mode shapes of vibration

Mode shapes are plotted to describe the characteristics of transverse and rotational deflections of the beam. The first mode of vibration is predominantly associated with transverse motion, whereas the third mode exhibits a greater contribution for rotational vibrations. The numerical results indicate that the beam vibrates in a similar manner, regardless of the soil stiffness (k_0) and degrees of variation (α, β) of the foundation. The mode shapes depend only on boundary conditions for a beam without foundation. They are affected by the foundation stiffness for a beam resting on a linear Winkler foundation. Finally, for a beam on a variable Winkler foundation, the mode shapes depend on the spatial variation of foundation stiffness.

4. CONCLUSION

The spectral element method has been employed to investigate the free vibrations of an Euler-Bernoulli beam resting on a variable Winkler elastic foundation. The natural frequencies were computed for different stiffness distributions: constant, linear, and parabolic. Conclusions that can be drawn from this work are:

- The results demonstrate that the spectral element method (SEM) accurately captures the free vibration behavior of beams resting on variable Winkler elastic foundations. Compared with the conventional finite element method (FEM), SEM exhibits faster convergence and achieves comparable or higher accuracy with significantly fewer elements, leading to improved computational efficiency.
- Furthermore, the study provides new parametric insights into the influence of stiffness variation intensity and distribution type (constant, linear, and parabolic) on modal characteristics, highlighting mode-dependent sensitivity effects.
- The natural frequencies are sensitive to mechanical properties and spatial distribution of the foundation stiffness along the beam length.
- As the soil stiffness increases, the nonlinear natural frequencies also increase for higher vibration modes. Among the considered distributions, the parabolic variation yields higher frequencies compared to the constant and linear cases.
- The foundation stiffness influences the modal characteristics of a beam by modifying the global stiffness distribution of the system. While its effect on natural frequencies is generally significant, its impact on mode shapes is often more subtle.
- The developed numerical programs, based on FEM and SEM formulations, are able to efficiently analyze the dynamic responses of beams on variable Winkler elastic foundations with spatially varying stiffness and various boundary conditions.

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