An Overview of Analytical Methods for FEM Analysis of Two-Dimensional Elastic Multibody Systems

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Abstract: The last decades have registered enhanced performance of machines and mechanisms, due to their higher working speed and increased forces applied. As a consequence, the elastic elements of the multibody systems used in the analysis of different mechanical systems have become more relevant, their elasticity manifesting more strongly and leading to unwanted phenomena, even to system destruction. Therefore, this type of problem has become of great interest to researchers in the field of applied mechanics. This article presents the results of a theoretical study conducted on several works about analytical methods used for FEM analysis of elastic multibody systems in planar motion.

Keywords: vibration, two-dimensional FE, Lagrange's equations, Gibbs-Appell's equation, Maggi's equation, Hamilton's equation, MBS, FEM.

1. INTRODUCTION

Numerous works have been conducted on the study of the dynamic behavior of mechanical systems by using the multibody system approach (MBS), which is successfully applied to analyze systems working at high speeds and subjected to increased forces, as well as to different parameters and operation conditions. Their behavior was proved to be greatly influenced by the elasticity of the elements composing the multibody systems, which cannot be neglected or ignored. At the same time, the current level of development of the finite element method (FEM), makes it a good candidate for being used in the analysis of elastic MBSs.

2. ANALYTICAL APPROACHS USED FOR OBTAINING EQUATIONS OF MOTION

A particular aspect necessary to be considered in dynamic analysis with FEM is the specific MBS framework on which the method is applied, the formulation of motion equations being the primary step. At this stage, new terms that correspond to the effects caused by the different types of accelerations appear in the motion laws. The challenge that emerges in modeling is how to choose the best approximations to produce equations of motion that depict reality as accurately as possible. These new terms stand for the specificity in formulating the equations. One way to obtain the equations of motion is by using analytical mechanics [1-3], because its techniques provide the highest degree of generality when addressing a MBS problem. Furthermore, a variety of stiff or elastic components with a wide range of properties must be considered while studying contemporary systems. The systems under study require consistent design and numerical modeling efforts and are by no means straightforward.

It is clear from these situations that, for applications, in terms of modeling, amount of computing operations required, and expenses incurred by simulations, any benefit a selected approach may provide is important and should be considered. To achieve effective algorithms and user-friendly software, finally, the equations of motion are to be obtained. The application of highly generic techniques is possible by using analytical mechanics theory. This allows for consistently handling various applications.

The procedures provided by analytical mechanics play an important role in the analysis [4-9]. The basic laws of mechanics can also be expressed in several analogous ways using this approach, with similar results. Researchers apply different approaches depending on their background and the specific problem under investigation, considering the advantages and drawbacks of each of them.

Most researchers nowadays use Lagrange's equations as standard methodology, mainly because it leverage relatively basic mechanical concepts that researchers are familiar with (energy, work, or momentum) and guarantees a reasonable degree of generality for many problems faced in industrial applications. As a result, it can accurately characterize the limitations that might occur. There are many references in the literature that tackle the use of Lagrange's equations for modeling and analysis of the dynamic behavior of mechanical systems. This work highlights articles [10-14]. The method for generating the equations of motion can be freely chosen thanks to analytical mechanics, which can offer significant benefits in dynamic studies for specific engineering applications and situations. Some researchers have acknowledged and taken advantage of these benefits, which have shortened analysis times and simplified modeling [15-17]. To guarantee an ideal approach to the analysis, it is obvious that the use of analogous approaches also prompted research into the potential of employing the best numerical analysis software. Naturally, the FEM is a key component of these techniques [18-20].

Appropriate numerical techniques are needed to support applications of the analogous formulas provided by analytical mechanics. These techniques are described in [21] and aid in lowering the expenses associated with numerical analysis. One useful technique for reducing the time spent on modeling such systems is symbolic formalism [22].

Elastic body deformations are overlaid on top of the general rigid movement of an MBS. In [23], a standardized attempt is made to make it easier to write algorithms in MBSs. MBS techniques are used in a wide range of fields [24]. In [25,26], models for resolving MBS challenges in many domains are provided. These systems are studied, and certain engineering applications are analyzed in [27,28]. The study of complex systems is necessary for the current applications already in use in industry, and modeling and simulating such systems require large investments in resources (human resources and infrastructure – both hardware and software). Therefore, the aim is to achieve a trade-off between modeling systems as simple as possible, at the same time obtaining results as accurately as possible. Over the past ten years, various methods for formulating motion equations symbolically and reducing the time for modeling have been developed. In [29,30], two approaches that provide effective models for a particular elastic MBS are discussed. [31,32] describe the classical approaches used in the MBS. Significant research has been conducted on this subject, with varying degrees of success [33]. The traditional methods for resolving these problems are described in [34], with distinct solutions for various applications [35]. In [36], the application of a composite material to be used for manufacturing an MBS is examined. Writing equations of motion to study a general MBS is the aim of papers [37–39].

Practical applications commonly use two-dimensional elements (many engineering systems being planar) [40]. [41] illustrates the use of two-dimensional thin plates in a real word problem, by examining the natural frequencies for system optimization. For a multibody system, the transfer matrix method was the modeling technique employed. The theoretical underpinnings of this approach were also discussed, along with some specific instances that supported the study.

In [42], an analysis using a bi-dimensional finite element is used for the study of an aircraft's wing skin. Vibrations were superimposed over the six degrees of freedom that the investigated plane had for rigid body motion. The study investigated how elasticity affected the total motion and decisions were taken throughout the plane's design phase based on that study. In [43], a shell element with revolving blade geometrical features was created. The suggested model was validated by comparing its predictions with other research results from the literature.

In [44], a model for massive structures was developed using FEM. The synthesis of a spacecraft's dynamic model demonstrated the benefits of the proposed model. There were two movable and flexible solar panels on the ship. The application of an FE model using two-dimensional shell finite elements was examined. In [45], a novel type of finite element featuring a thin hyper-elastic shell was introduced. The Kirchhoff–Love theory served as the foundation for the idea. In [46,47], the Gibbs–Appell (GA) approach was introduced for the use of two-dimensional finite elements. Additional intriguing findings can be found in [48–53].

There are many difficulties in the application of models for planar structures or MBS in the industrial area; therefore, a thorough analysis of these problems is necessary. By carefully examining the primary approaches provided by analytical mechanics in its classic form, this research draws attention to models used for analyzing planar structures or MBS with elastic elements by using the finite element method, more specifically the two-dimensional finite element model, considering its planar motion.

3. CONCEPTS AND MODEL

3.1. Basic Kinematics

In the dynamic analysis of MBS with elastic elements, the problem is to use appropriate models that accurately capture the factors that influence the response of the system. FEM is currently a method acknowledged by engineers and researchers to solve such applications. It implies the use of the fundamental principles of mechanics to determine the system's evolution in time. To achieve this, although several methods are used, the established method remains the Method of Lagrange's Equations (LE). The practical applications of recent years, which involve complex systems with large dimensions but which also present certain symmetries or peculiarities, led to the conclusion that other methods known from analytical mechanics could be more advantageous for the user in obtaining the equations of motion [54]. Thus, this paragraph systematically presents some methods that can substitute the Method of LE and can be selected to be used in specific problem solving, when their advantages are obvious.

To carry out this analysis, a certain two-dimensional finite element is considered and characterized from the dynamic point of view in terms of energy, work, and momentum. The finite element is related to a local, mobile reference frame, related to the finite element, *Oxy,* having unit vectors $\bar{i}$ and $\bar{j}$. The reference frame contributes, together with the finite element, to the "rigid motion" of the part of the studied mechanical system. The velocity of the origin of this reference system is $\bar{v}_o(v_{O1}, v_{O2}j)$. The acceleration of the origin is $\bar{a}_o(a_{O1}, a_{O2}j)$, the angular velocity in which the mobile reference system is driven is $\overline{\omega} = \omega \bar{k}$ and the angular acceleration is $\bar{\varepsilon} = \varepsilon \bar{k}$. The last two vectors are oriented perpendicularly on the plane of motion. It is assumed that these four vectors that define the motion of the mobile reference frame are determined in a previous study of MBS, which is considered a system with rigid elements [16-18]. It is assumed that the deformation of the elements is in the plane of motion.

Index L is used to specify the local (mobile) coordinate system and G for the global coordinate system. With θ is denoted the angle of rotation. If the components of a vector are known in the L reference system, they can be calculated in the G reference frame with the relation [16]:

$$\{v\}_{Ox_1y_1} = [R]\{v\}_{Oxy} \tag{1}$$

where the rotation matrix is:

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

The following notations are used:

- $\bar{r}_O$ is the position vector of the origin O of the L reference frame related to the G coordinate system having the origin in O_1. The components in the G are $(X_{O,1}, X_{O,2})$ and in the L are $(x_{O,1}, x_{O,2})$.
- $\bar{r}_M$ is the position vector of an arbitrary point M. The components in the G are $(X_{M,1}, X_{M,2})$ and $(x_{M,1}, x_{M,2})$ in L.
- $\bar{r}_{M'}$ is the position of M after displacement/deformation when it becomes *M'* with the components in L respectively $(x_{M',1}, x_{M',2})$. The components of the vector $\bar{r}$ of point M with respect to origin O are (X_1, X_2), respectively (x_1, x_2), in the two reference frames and the displacement vector $\bar{u} = \overline{MM}'$, has the components (u, v).

The elastic finite element deforms. So an arbitrary point *M* becomes the point *M'* and its coordinates can be written as:

$$X_{M',1} = X_{O,1} + (x_1 + u)\cos\theta - (x_2 + v)\sin\theta \quad ;$$
$$X_{M',2} = X_{O,2} + (x_1 + u)\sin\theta + (x_2 + v)\cos\theta . \tag{3}$$

The coordinates $(X_{M',1}, X_{M',2})$ are expressed in the *G* reference frame while $(x_{M',1}, x_{M',2})$ are in the *L* reference frame.

3.2. Basic concepts

3.2.1. Kinetic Energy

The kinetic energy accumulated in a finite element is [18]:

$$E_C = \frac{1}{2}\int_V \rho v_M^2 dV = \frac{1}{2}\int_V \rho\left[(\dot{x}_{M',1})^2 + (\dot{x}_{M',2})^2\right]dV \tag{4}$$

The following notions are used:

$$m = \int_V \rho dV \; ; J_O = \int_V \rho\left(x_1^2 + x_2^2\right)dV \; ;$$

$$m_{rt} = \int_V \rho N_{kr} N_{kt} dV \;\; ; \; m_{1,mr} = \int_V \rho x_1 N_{mr} dV \;\; ;$$

$$S_1 = \int_V \rho x_1 dV \quad ; \quad S_2 = \int_V \rho x_2 dV \;\; ;$$

$$m^I_{O,kr} = \int_V \rho N_{kr} dV \quad ; \quad m_{1,mr} = \int_V \rho x_1 N_{mr} dV \quad ;$$

$$m_{2,mr} = \int_V \rho x_2 N_{mr} dV \quad . \tag{5}$$

Thus, the kinetic energy for a single finite element can be written [57]:

$$E_c = \frac{1}{2} m\left(\dot{x}^2_{O,1} + \dot{x}^2_{O,1}\right) + \frac{1}{2}\omega^2 J_O$$
$$+ \frac{1}{2}\omega^2 \delta_t \delta_r m_{rt} + \frac{1}{2}\dot{\delta}_r \dot{\delta}_t m_{rt} - \omega\left(\dot{x}_{O,1} S_2 - \dot{x}_{O,2} S_2\right)$$
$$- \omega\delta_r\left(\dot{x}_{O,1} m^I_{O,2r} + \dot{x}_{O,2} m^I_{O,1r}\right) + \omega^2 \delta_r\left(m_{2,2r} + m_{1,1r}\right)$$
$$+ \left(\dot{x}_{O,1} m^I_{O,1r} + \dot{x}_{O,2}\,\dot{\delta}_r m^I_{O,2r}\right)\dot{\delta}_r - \omega\dot{\delta}_r\left(m_{2,1r} - m_{1,2r}\right)$$
$$- \omega\delta_r \dot{\delta}_t\left(m_{12,rt} - m_{21,rt}\right) \quad . \tag{6}$$

The material density is denoted with ρ and it is considered constant for a finite element.

3.2.2. Potential Energy

The potential energy can be written in this case as [16,18,57]:

$$E_p = \frac{1}{2}\int_V \left(\sigma_{11}\varepsilon_{11} + 2\sigma_{12}\varepsilon_{12} + \sigma_{22}\varepsilon_{22}\right) dV \tag{7}$$

where:

$$\sigma_{11} = \frac{E}{1-2\mu}\varepsilon_{11} \quad ; \quad \sigma_{22} = \frac{E}{1-2\mu}\varepsilon_{22} \quad ;$$
$$\sigma_{12} = \frac{E}{2(1-\mu)}\varepsilon_{12} \tag{8}$$

The strain can be obtained using the relation:

$$\varepsilon_{11} = \frac{\partial u}{\partial x_1} \quad ; \quad \varepsilon_{22} = \frac{\partial v}{\partial x_2} \quad ; \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right) \tag{9}$$

In this analysis, if the shape functions N are considered the above-mentioned relations become:

$$\varepsilon_{11} = \frac{\partial u}{\partial x_1} = \frac{\partial N_{1r}}{\partial x_1}\delta_r ; \; \varepsilon_{22} = \frac{\partial v}{\partial x_2} = \frac{\partial N_{2r}}{\partial x_2}\delta_r ;$$
$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial N_{1r}}{\partial x_2} + \frac{\partial N_{2r}}{\partial x_1}\right)\delta_r \tag{10}$$

and the stresses are:

$$\sigma_{11} = \frac{E}{1-2\mu}\frac{\partial N_{1r}}{\partial x_1}\delta_r ; \; \sigma_{22} = \frac{E}{1-2\mu}\frac{\partial N_{2r}}{\partial x_2}\delta_r ;$$
$$\sigma_{12} = \frac{E}{4(1-\mu)}\left(\frac{\partial N_{1r}}{\partial x_2} + \frac{\partial N_{2r}}{\partial x_1}\right)\delta_r \quad . \tag{11}$$

Using these relations, the potential energy can be written as:

$$E_p = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij} dV = \frac{1}{2} k_{rt}\delta_r\delta_t \quad , \tag{12}$$

where the elements k_{rt} of the stiffness matrix are obtained via the relations [57]:

$$k_{rt} = \frac{1}{2}\int_V \left[\frac{E}{1-2\mu}\frac{\partial N_{1r}}{\partial x_1}\frac{\partial N_{1t}}{\partial x_1} + \frac{E}{1-2\mu}\frac{\partial N_{2r}}{\partial x_2}\frac{\partial N_{2t}}{\partial x_2}\right]$$
$$+ \frac{1}{2}\int_V \left[\frac{E}{8(1-\mu)}\left(\frac{\partial N_{1r}}{\partial x_2} + \frac{\partial N_{2r}}{\partial x_1}\right)\left(\frac{\partial N_{1t}}{\partial x_2} + \frac{\partial N_{2t}}{\partial x_1}\right)\right] \tag{13}$$

3.2.3. Work

The work produced by the generalized concentrated forces $i = \overline{1,p}$ is [16,18,57]:

$$W^c = q_i \delta_i \quad ; \quad i = \overline{1,p} , \tag{14}$$

and the work produced by the generalized volume forces q^*_i $i = \overline{1,p}$, is:

$$W^{\mathrm{d}} = q^*_i \delta_i \quad ; \quad i = \overline{1,p} \quad . \tag{15}$$

So, the total work can be written as:

$$W = \left(W^c + W^{\mathrm{d}}\right) = \left(q_i + q^*_i\right)\delta_i \quad ; \quad i = \overline{1,p} \quad . \tag{16}$$

3.2.4. Lagrangian

The standard expression for the Lagrangian is:

$$L = E_c - E_p + W \quad . \tag{17}$$

Equations (6), (12), and (16) are used in (17) to obtain the Lagrangian [46,47]:

$$L = \frac{1}{2} m\left(\dot{x}^2_{O,1} + \dot{x}^2_{O,2}\right) + \frac{1}{2}\omega^2 J_O$$
$$+ \frac{1}{2}\omega^2 \delta_t \delta_r m_{rt} + \frac{1}{2}\dot{\delta}_r \dot{\delta}_t m_{rt}$$
$$- \omega\left(\dot{x}_{O,1} S_2 - \dot{x}_{O,2} S_1\right) - \omega\delta_r\left(\dot{x}_O m^I_{O,2r} + \dot{x}_{O,2} m^I_{O,1r}\right)$$
$$+ \left(\dot{x}_{O,1} m^I_{O,1r} + \dot{x}_{O,2}\,\dot{\delta}_r m^I_{O,2r}\right)\dot{\delta}_r + \omega^2 \delta_r\left(m_{2,2r} + m_{1,1r}\right)$$
$$- \omega\dot{\delta}_r\left(m_{2,1r} - m_{1,2r}\right) - \omega\delta_r\dot{\delta}_t\left(m_{12,rt} - m_{21,rt}\right)$$
$$- k_{rt}\delta_r\delta_t + q_r\delta_r + q^*_r\delta_r \quad ; \quad r,t = \overline{1,p} \quad . \tag{18}$$

3.2.5. Momentum

The momentum can be obtained using the relation:

$$p_{r,L} = \frac{\partial L}{\partial \dot{\delta}_r} \tag{19}$$

It results [46,47]:

$$p_{r,L} = \frac{\partial L}{\partial \dot{\delta}_r} = m_{rt}\dot{\delta}_t + \left(\dot{x}_O m^I_{O,1r} + \dot{y}_O m^I_{O,2r}\right)$$
$$- \omega\left(m_{y,1r} - m_{x,2r}\right) - \omega\left(m_{xy,rt} - m_{yx,rt}\right)\delta_r ;$$
$$r,t = \overline{1,p} \quad . \tag{20}$$

The inverse-matrix of m_{rt}, denoted m^*_{ur}. The following relation is considered:

$$m^*_{ur} m_{rt} = \delta_{ut} \quad ; \quad u,r,t = \overline{1,p} \quad , \tag{21}$$

where δ_{ut} is the Kronecker delta.

Pre-multiplying Equation (28) with m^*_{ur} it yields:

$$\begin{aligned} \dot{\delta}_u &= m^*_{ur} p_{r,L} - m^*_{ur}\left(\dot{x}_O m^I_{O,1r} + \dot{y}_O m^I_{O,2r}\right) \\ &+ \omega m^*_{ur}\left(m_{y,1r} - m_{x,2r}\right) + \omega m^*_{ur}\left(m_{xy,rt} - m_{yx,rt}\right)\delta_r \\ & u,r,t = \overline{1,p} \end{aligned} \tag{22}$$

which represents the velocity of the generalized coordinate, written in terms of generalized coordinates and generalized momentum.

3.2.6. Hamiltonian

The Hamiltonian is [46,47]:

$$H = \frac{\partial L}{\partial \dot{\delta}_r}\dot{\delta}_r - L \quad . \tag{23}$$

By using Eq. (18) and Eq. (22) it is possible to obtain the final form for Eq.(23).

3.2.7. Energy of Accelerations

The GA equation is a significant technique provided as a substitute form for Lagrange's equations. The energy of acceleration is used by this method. The energy of acceleration is, in our case [10,18]:

$$E_a = \frac{1}{2}\int_V \rho a^2 dV \quad ; \tag{24}$$

$$\begin{aligned} \ddot{x}_{M',1} &= \ddot{x}_{O,1} + \left(x_1 + N_{1j}\ddot{\delta}_j\right) - \omega^2\left(x_1 + N_{1j}\delta_j\right) \\ &- \varepsilon\left(x_2 + N_{2j}\delta_j\right) - 2\omega\left(x_2 + N_{2j}\dot{\delta}_j\right) \quad ; \\ \ddot{x}_{M',2} &= \ddot{x}_{O,2} + \left(x_2 + N_{2j}\ddot{\delta}_j\right) - \omega^2\left(x_2 + N_{2j}\delta_j\right) \\ &+ \varepsilon\left(x_1 + N_{1j}\delta_j\right) + 2\omega\left(x_1 + N_{1j}\dot{\delta}_j\right) \quad , \end{aligned} \tag{25}$$

where the acceleration is:

$$a^2 = \ddot{x}_{M',1} + \ddot{x}_{M',2} \quad . \tag{26}$$

Introducing this expression in Eq. (24) it is possible to obtain the final form for the energy of acceleration.

4. ALTERNATIVE FORMULATIONS

Analytical mechanics offers a series of equivalent formalisms for obtaining the equations of motion for any mechanical system. This paragraph presents the main formalisms applied in the MBS with elastic elements where FEM is used. Other formalisms, more seldom used, are essentially based on those presented below.

4.1. Lagrange's Equations

When faced with challenges like those in an MBS, the application of Lagrange's equations proved beneficial. Using scalars rather than vectors proved to be an advantage. The formulas of kinetic and potential energy were used to determine the generalized forces using Lagrange's equations [55,56]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}_i}\right) - \frac{\partial L}{\partial \delta_i} = 0 \quad ; \quad i = \overline{1,p} \tag{27}$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{\delta}_r} &= \dot{\delta}_t m_{rt} + \left(\dot{x}_{O,1} m^I_{O,1r} + \dot{x}_{O,2} m^I_{O,2r}\right) \\ &- \omega\left(m_{2,1r} - m_{1,2r}\right) - \omega\delta_r\left(m_{12,rt} - m_{21,rt}\right) \quad ; \quad r,t = \overline{1,p} \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\delta}_r} &= \ddot{\delta}_t m_{rt} + \left(\ddot{x}_{O,1} m^I_{O,1r} + \ddot{x}_{O,2} m^I_{O,2r}\right) \\ &- \varepsilon\left(m_{2,1r} - m_{1,2r}\right) - \varepsilon\delta_r\left(m_{12,rt} - m_{21,rt}\right) \\ &- \omega\dot{\delta}_r\left(m_{12,rt} - m_{21,rt}\right) \quad ; \quad r,t = \overline{1,p} \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\partial L}{\partial \delta_r} &= \omega^2 \delta_t m_{rt} - \omega\left(\dot{x}_{O,1} m^I_{O,2r} + \dot{x}_{O,2} m^I_{O,1r}\right) \\ &+ \omega^2\left(m_{2,2r} + m_{1,1r}\right) - \omega\dot{\delta}_t\left(m_{12,rt} - m_{21,rt}\right) \\ &- k_{rt}\delta_t + q_r + q^*_r \quad ; \quad r,t = \overline{1,p} \end{aligned} \tag{30}$$

By introducing Eq. (29) and Eq. (30) in Eq. (27) the equation of motion is obtained:

$$\begin{aligned} & m_{rt}\ddot{\delta}_t + 2\omega\dot{\delta}_r\left(m_{12,rt} - m_{21,rt}\right) \\ &+ \left[k_{rt} - \varepsilon\left(m_{12,rt} - m_{21,rt}\right) - \omega^2 \delta_t m_{rt}\right]\delta_t \\ &= -\left(\ddot{x}_{O,1} m^I_{O,1r} + \ddot{x}_{O,2} m^I_{O,2r}\right) + \varepsilon\left(m_{2,1r} - m_{1,2r}\right) \\ &- \omega\left(\dot{x}_{O,1} m^I_{O,2r} + \dot{x}_{O,2} m^I_{O,1r}\right) + \omega^2\left(m_{2,2r} + m_{1,1r}\right) \\ &+ q_r + q^*_r \quad ; \quad r,t = \overline{1,p} \end{aligned} \tag{31}$$

4.2. Hamilton's Method

In Section (4.1) a second-order differential equations system was obtained to determine the evolution equations of the MBS using Lagrange's method. This system needs to be converted into a first-order differential equations system by adding new unknown variables, in order to be solved. Therefore, the dimension of the differential equation doubles. When the Hamilton approach is applied, the first-order equations are directly derived but with double dimensions. The generalized coordinates and the canonical conjugate momenta are the unknown variables in this formulation [54,57]:

$$p_{i,L} = \frac{\partial L}{\partial \delta_i} \quad ; \quad i = \overline{1,p} \ . \qquad (32)$$

Thus, the use of canonical conjugated momenta rather than generalized velocities is the primary distinction between Lagrange's and Hamilton's approaches. The primary benefit of using these equations is that a first-order equations system could be obtained straight away and solved directly, as first-order differential equations are easy to solve (given the wide offer of software available). The traditional Hamilton's equations are:

$$\dot{\delta}_r = \frac{\partial H}{\partial p_{r,L}} \quad ; \quad \dot{p}_{r,L} = -\frac{\partial H}{\delta_r} \ . \qquad (33)$$

By using Eq. (23), the above-mentioned relations become [54,57]:

$$\dot{\delta}_r = m^*_{ru} p_{u,L} - m^*_{ru} m_{O,iu} \dot{x}_{O,i} - m^*_{ru} m_{k,iu} \alpha_{ij} \dot{\alpha}_{jk} - m^*_{ru} m_{ik,ut} \alpha_{ij} \dot{\alpha}_{jk} \delta_t \quad ; \quad u,r,t = \overline{1,p} \qquad (34)$$

$$\dot{p}_{r,L} = \frac{\partial L}{\partial \delta_r} = \omega^2 \delta_t m_{rt} - \omega\left(\dot{x}_{O,2} m^I_{O,2r} + \dot{x}_{O,2} m^I_{O,1r}\right) + \omega^2\left(m_{2,2r} + m_{1,1r}\right) - \omega\dot{\delta}_t\left(m_{12,rt} - m_{21,rt}\right) - k_{rt}\delta_t + q_r + q^*_r \quad ; \quad r,t = \overline{1,p} \qquad (35)$$

4.3. Gibbs–Appell Equation

Currently, the GA equation formalism is reexamined considering the requirement to evaluate an MBS with elastic elements. The acceleration energy, which is given in Eq. (33), is the primary concept utilized in GA equations. The GA equation has the following form [54]:

$$\frac{\partial E_a}{\partial \ddot{\delta}_r} = Q_r \, , \quad r = \overline{1,p} \ , \qquad (36)$$

where:

$$Q_r = q_r + q^*_r \ . \qquad (37)$$

The total kinetic energy can be written as:

$$E_a = E_{ao} + E_{a1} + E_{a2} \ , \qquad (38)$$

where E_{ao} has no terms containing generalized acceleration, E_{a1} contains the terms with linear terms in accelerations and E_{a2} contains terms with quadratic terms in acceleration.

$$E_{a1} = \ddot{x}_{O,i} \ddot{\delta}_r m^I_{O,ir} + \left(\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ij}\ddot{\alpha}_{jk}\right)\ddot{\delta}_r m_{k,mr} + \left(\dot{\alpha}_{ji}\dot{\alpha}_{jk} + \alpha_{ij}\ddot{\alpha}_{jk}\right)\delta_r \ddot{\delta}_t m_{kr,mt} + 2\alpha_{ji}\dot{\alpha}_{jk}\dot{\delta}_r\ddot{\delta}_t m_{kr,mt} dV \qquad (39)$$

$$E_{a2} = \frac{1}{2} m_{rt} \ddot{\delta}_r \ddot{\delta}_t \quad r,t = \overline{1,p} \ . \qquad (40)$$

It results:

$$\frac{\partial\left(E_{a1} + E_{a2}\right)}{\partial \ddot{\delta}_r} = Q_r \, , \quad r = \overline{1,p} \quad ; \qquad (41)$$

$$Q_{r,L} = k_{rt}\delta_t + q_r + q^*_r \quad ; \quad r,t = \overline{1,p} \, . \qquad (42)$$

It is noted that this approach uses fewer differentials compared to Lagrange's equations. As a result, the modeling and simulation processes take less time and cost less.

4.4. Maggi's Equation

Refs. [54] explains the use of Maggi's equation and how they are applied to solve related problems. Maggi's equations are represented by the set of $p = N\text{-}m$ independent equations. The system's generalized independent coordinates are $q_1, q_2, \ldots, q_N$; there are m linear relations between them, with the form [54]:

$$a_{ij}(q_1, q_2, \ldots \ , q_N, t)\dot{q}_j + b_i(q_1, q_2, \ldots \ , q_N, t) = 0 \qquad i = \overline{1,m} \, ; \quad j = \overline{1,N} \ . \qquad (43)$$

$$a_{kj}\left[\left(\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k}\right) - Q_k\right] = 0 \quad ; \qquad j = \overline{1,N-m} \, ; \quad k = \overline{1,N} \qquad (44)$$

Equation (40) is ultimately obtained by applying these equations to a single finite element.

4.5. Kane's Equations

By using the relation [54,57]:

$$\left(\overline{F}_i - m_i \overline{a}_i\right)\delta\overline{r}_i = 0 \quad , \quad i = \overline{1,N} \ , \qquad (44)$$

true for a system of N degrees of freedom, (defined by a number of p generalized coordinates), it yields

$$\sum_{i=1}^{N}\left(\overline{F}_i - m_i \overline{a}_i\right)\frac{\partial \overline{r}_i}{\partial q_k} = 0 \quad ; \quad k = \overline{1,p} \qquad (45)$$

$\overline{F}_i$ being the external forces acting in the nodes.

By using the relation [54]:

$$\frac{\partial \overline{r}_i}{\partial q_k} = \frac{\partial \overline{v}_i}{\partial \dot{q}_k} \quad ; \quad k = \overline{1,p} \quad , \qquad (46)$$

in Eq. (45), it yields:

$$\sum_{i=1}^{N}\left(\overline{F}_i - m_i \overline{a}_i\right)\frac{\partial \overline{v}_i}{\partial \dot{q}_k} = 0 \quad ; \quad k = \overline{1,p} \ . \qquad (47)$$

Considering the notation:

$$\frac{\partial \overline{v}_i}{\partial \dot{q}_k} = \frac{\partial \overline{v}_i}{\partial u_k} = \overline{v}^{(k)}_i \quad ; k = \overline{1,p} \quad ; i = \overline{1,N} \ , \qquad (48)$$

Eq. (47) becomes:

$$\sum_{i=1}^{N}\overline{F}_i\ \frac{\partial\overline{v}_i}{\partial u_k}=\sum_{i=1}^{N}m_i\overline{a}_i\,\frac{\partial\overline{v}_i}{\partial u_k};\quad k=\overline{1,p}\,;\quad i=\overline{1,N}\,. \tag{49}$$

For an elastic finite element considered as a solid, Eq. (60) becomes:

$$\sum_{i=1}^{N}\overline{F}_i\frac{\partial\overline{v}_i}{\partial\dot{q}_k}=\int_V\overline{a}\frac{\partial\overline{v}}{\partial\dot{q}_k}dm \qquad k=\overline{1,p} \tag{50}$$

Thus, after some calculations, it was possible to obtain Eq. (31) using this alternative approach.

5. CONCLUSIONS AND DISCUSSION

An increasingly popular technique in the analysis of highly complex industrial applications is to use FEM in the dynamic analysis of elastic MBS. The intricacy and unique characteristics of the system under study make it challenging to derive the evolution equations. These can be obtained with the help of several equivalent classic formalisms.

Since analyzing MBS with planar motion is quite usual in industrial applications, there is a high interest in studying these mechanical systems more thoroughly. This research studies how alternative techniques of analytical mechanics are used to model and describe this type of system. The Lagrange, GA, Hamilton, Kane, and Maggi formalism have been considered for discussion. Of course, the approaches described are interchangeable, and it is the researchers to say which approach would work best for a specific application, based on their background and prior experience.

Most researchers use nowadays Lagrange's equations. In chapter 2 of this work, the reasons have been explained: the simplicity of the method, the high degree of generality, the ease of including it in an algorithm, and the familiarization with the methodology and basic concepts it uses.

The benefit of using fewer differentiation processes was anticipated by GA's equations. The modeling and calculation of such a system imply lower time and costs for designers. The primary challenge was applying the idea of the energy of acceleration since researchers are less familiar with it. However, the need for efficient techniques to analyze complex MBS makes the GA approach more and more interesting and considered in the modeling process.

The Lagrangian formalism is basically developed into Maggi's equation. The application of this method is highly suitable for the research of non-holonomic systems. The basic equations of motion can be readily obtained by knowing the values of mechanical energies as well as the connections in the finite element network (kinematic conditions). Additionally, the binding forces elimination techniques empirically used in FEM software are justified by this formalism.

Maggi's equation method, which is the source of Kane's formalism, is extremely similar to it. Industrial robot applications and the automation sector have recently made use of it. An effective substitute that has the benefit of being cost-effective when studying systems with a high number of degrees of freedom is Kane's equations. This approach is a logical substitute for non-holonomic systems. The necessity of dealing with intricate mechanical systems that function under challenging conditions and at high speeds explains the interest in alternative ways of describing their behavior, including the use of Kane's method.

Researchers are beginning to revisit Hamilton's equations considering the current technological environment. They are quite basic and have a high degree of generality. The Hamiltonian is defined using generalized coordinates and their conjugate moments. Furthermore, first-order differential equations are used instead of second-order equations resulting from other techniques.

It can be therefore concluded that alternative analytical mechanics approaches should be considered for modeling and analysis of complex systems. Due to the benefits proven in specific applications that require time-consuming numerical analyses, they could be interesting candidates to address the challenges induced by the present technological environment.

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