# Particularities of the Modal Analysis in the Study of Flexible Multibody Systems

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*Abstract:* - Modal analysis represents a powerful method to analyze the vibration of various types of mechanical systems. Using this method in the classic way is possible with certain approximations. In this paper, we propose an extension of this method for studying a multibody system (MBS) with flexible elements. The particular shape of the equations of motion, acquired for these mechanical systems, imposes precautions in the use of the method. We highlight the particularities of the equations that enable the use of the results obtained in the classic modal analysis.

*Keywords:* - vibration, Lagrange equations, MBS, finite element method, modal analysis

## 1. INTRODUCTION

Modal analysis has proven to be a powerful tool for studying linear elastic systems. There are many papers that prove the efficiency and usefulness of this method. In the following, some suggestive results from extensive research conducted in recent decades will be presented. They stand out for the diversity of the subjects covered and the fields in which this analysis can be useful.

A complex work that combines the transfer matrix method for flexible (MBS) structures with the finite element method (FEM) allows the eigenvalues of such problems to be obtained [1]. A comparison with the application of only the FEM shows a very good match. The advantages include ease of programming, low computational effort, and, therefore, low costs when addressing such problems. The analysis of a dynamic railway locomotive model using modal analysis techniques is presented in [2]. The dynamic loads acting on the bogie frame are determined using MBS analysis. The software ANSYS was used to model and analyze the bogie frame. A detailed study of gearbox behavior using modal analysis is presented in [3]. The study of the gearbox assembly is preceded by an analysis of each part, separately from the gearbox, studying its vibration behavior. The strength and fatigue limits of a car body are determined using the modal superposition method in [4]. Various domains benefit from modal analysis, which helps solve complex problems in engineering applications [5-10]. A trend in the design of mechanical systems is to reduce component weight. This would help reduce the energy required for operation and enable higher speeds. However, these advantages are useless if the structure does not exhibit good vibration behavior. To prevent vibration problems from becoming critical for the structure, it is necessary to study the eigenfrequencies. Such a study in this direction is done in [11]. A parametric modal model allows obtaining a Taylor series expansion in the vicinity of a chosen configuration and selecting a system configuration that yields the desired eigenfrequency in operation.

An interesting paper that develops a model to ensure high computational efficiency presents a new transfer matrix method for studying the linear vibrations of flexible multibody systems [12]. In the form in which the dynamic equations are written, they can also provide the answer to known external excitations. The proposed method is verified by studying the dynamics of an ultra-precision machine tool with flexible elements. The advantages include the ease of deducing the transfer equation and the higher calculation speed. and the ability to quickly obtain results.

Modal analysis for flexible MBS has been applied in many studies, but only in the classic form, which involves simplifying the equations of motion. For example, in [13], it is proposed to use motion modes to create flexible algorithms for the dynamic analysis of MBS. FEM is used as an auxiliary element in the proposed method. The paper's results are verified in a real-world application, the Shuttle Remote Manipulator, which represents the MBS system to which the developed methodology was applied. In the manufacturing industry, digital twins of machine tools are currently used to validate and optimize execution processes. The limited computational capabilities and low computational efficiency of these systems make it difficult to approach MBS problems using flexible elements. An analytical model of a flexible MBS system [14] enables analysis of a process (the complex movement of the MBS) two times faster than the actual processing, allowing real-time decision-making. The calculations for a horizontal machine tool yielded an average accuracy of 88%, which is very good for the studied application. A review of evolution in the study of flexible MBS is done in [15]. The basic approaches in the field, focusing on the kinematics and dynamics of these systems, are presented, and future research directions are identified. The main direction of study is the use of incremental methods with finite elements, which also includes the present work. The study presents linearization methods for the MBS motion equations. The solution procedures constitute a separate chapter of the research in the field, with less importance for the engineer who proposes the calculation models, but are essential for product developers. Each method or formulation proposed for solving this type of problem involves approximations and assumptions that may be simplified and not yet sufficiently discussed. In [16], elastic structural behavior is accounted for in the development of the models used by MBS software packages. They are characterized by significant stiffening effects in some elements of an MBS's composition. The results obtained are illustrated with an example. Such developments are key to simulating structures. Several special problems arising from flexible MBS analysis are presented in [17-25].

Experimental modal analysis is a widely used method for analyzing dynamic systems, with numerous studies illustrating its application [26]-[32].

In what follows, the modal analysis method is applied to the analysis of flexible MBS. The obtained equations of motion are second-order differential equations with constant coefficients over very small-time intervals. Compared with the classic systems of equations obtained in MBS studies, these additional terms prevent the direct application of modal analysis. Thus, a symmetric skew matrix appears in the position of the viscous damping matrix, which is symmetric. Also, the stiffness matrix is modified by the change in the finite element geometry during the movement of two matrix terms. One is symmetric, and the other is skew-symmetric. The paper will also describe the application of modal analysis in the study of MBS.

## 2. MOTION EQUATIONS

In the FEM analysis of a flexible MBS, an important step is obtain the equations of motion for a finite element. For a one-dimensional element, the problem is well studied (see, for example, [33]-[40]).

The bodies composing an MBS are supposed to be linear elastic. The unknowns are represented by the independent nodal coordinates. Compared with classical FEM applications, MBS analysis yields additional terms. These additional terms arise from the relative motion of the local reference frame (LRF) with respect to the global reference frame (GRF) due to Coriolis effects and from changes in stiffness, a phenomenon associated with the acceleration field in rigid motion. Additional inertial terms also appear due to the motion of the finite element in relation to the LRF/GRF. Once the motion equations are determined, they must be applied to all elements relative to the GRF using the corresponding rotation matrix. The next step is to assemble all the equations to obtain the motion equations for the entire system in terms of nodal coordinates.

Basic results concerning the kinematics and dynamics of the finite element are presented below. Oxyz represents the LRF, and OXYZ the GRF. The Lagrange equations will be used to determine the motion equations [41,42].

If we consider a single finite element in the LRF, the velocity of an arbitrary point, *M,* that becomes after deformation *M',* is [43]:

$$\begin{aligned}\{v_{M'}\}_G &= \{\dot{r}_{M'}\}_G \\ &= \{\dot{r}_O\}_G + [\dot{R}]\{r\}_L + [\dot{R}][N]\{\delta_e\}_L + [R][N]\{\dot{\delta}_e\}_L\end{aligned} \tag{1}$$

where :

- with the index *L/G* is noted a size (vector, matrix) expressed in the LRF/GRF;

- $\{v_{M'}\}_G = \{\dot{r}_{M'}\}_G / \{v_{M'}\}_L = \{\dot{r}_{M'}\}_L$ are the velocity vectors with components expressed in the GRF/LRF;
- $\{v_O\}_G = \{\dot{r}_O\}_G$ represents the velocity of the origin of the LRF expressed in the GRF;
- $[R]$ represents the rotation matrix (his role is to transform the vector from the LRF to the GRF);
- $[N]$ is the matrix of the shape functions. This matrix depends on the type of chosen finite element;
- $\{r\}_L$ represent is the position vector of a current point of a finite element expressed in the LRF;
- $\{\delta_e\}$ is the vector of the nodal coordinates expressed in the LRF;
- $\{\dot{\delta}_e\}$ is the vector of the nodal generalized velocity with components in LRF.

With the presented notation, the kinetic energy can be written as:

$$E_c = \frac{1}{2}\int_V \rho \{v_{M'}\}_G^T \{v_{M'}\}_G dV \; . \tag{2}$$

The notation $\rho$ is for the material density of the finite element. The potential energy is:

$$E_p = \frac{1}{2}\int_V \{\delta_e\}^T [k] \{\delta_e\} dV \; . \tag{3}$$

where $[k]$ is the rigidity matrix [44]. Concentrated forces $\{q\}_L$ applied in the nodes of the finite element gives the work:

$$W^c = \{q\}_L^T \{\delta_e\}, \tag{4}$$

The volume forces $\{p\} = \{p(x, y, z)\}$, reduced in the nodal points, gives the work:

$$\begin{aligned} W^d &= \int_V \{p\}_L^T \{f\}_L dV \\ &= \left( \int_V \{p\}_L^T [N] dV \right) \{\delta_e\} = \{q^*\}_L^T \{\delta_e\} \end{aligned} \; . \tag{5}$$

where $\{f\}_L$ represents the displacement vector. The Lagrangian for a finite element is:

$$L = E_c - E_p + W \; . \tag{6}$$

where: $W = W^c + W^d$ is the total work. Lagrange equations, for a finite element, are [45-48]:

$$\frac{d}{dt}\left\{ \frac{\partial L}{\partial \{\dot{\delta}_e\}} \right\} - \left\{ \frac{\partial L}{\partial \{\delta_e\}} \right\} = 0 \tag{7}$$

Finally, the motion equations are:

$$\begin{aligned} &[m_e]\{\ddot{\delta}_e\} + [c_e]\{\dot{\delta}_e\} + ([k] + [k(\varepsilon)] + [k(\omega)])\{\delta_e\} \\ &= \{q_e^c\}_L + \{q_e^d\}_L - \{q_e^i(\varepsilon)\}_L - \{q_e^i(\omega)\}_L - [m_{Oe}^i]\{\ddot{r}_O\}_L \end{aligned} \tag{8}$$

To this level, the equations of motion are written in the LRF. They contain some sizes that change with the position of the bodies, but can be linearized considering the LRF "frozen" and the known field of velocities and accelerations being constant in a short period of time $\Delta t$ .

The motion equations (8) are expressed in the LRF. The coordinates of the nodes are expressed in this coordinate system. To provide a unitary description of the body's evolution, all coordinates may be expressed in the same coordinate system, GRF/LRF. For a single finite element *e,* the relation that made this transformation is in the form:

$$\{\Delta_e\} = [T_e]\{\delta_e\} \tag{9}$$

that offers:

$$\{\dot{\Delta}_e\} = [T_e]\{\dot{\delta}_e\} \quad \text{and} \quad \{\ddot{\Delta}_e\} = [T_e]\{\ddot{\delta}_e\} \; . \tag{14}$$

Premultiplying the system (12) with $[T_e]^T$ the results:

$$\begin{aligned} &[T]^T[m_e][T]\{\ddot{\Delta}_e\} + [T]^T[c_e][T]\{\dot{\Delta}_e\} + [T]^T([k] + [k(\varepsilon)] + [k(\omega)])[T]\{\Delta_e\} \\ &= [T]^T\{q_e^c\}_L + [T]^T\{q_e^d\}_L - [T]^T\{q_e^i(\varepsilon)\}_L - [T]^T\{q_e^i(\omega)\}_L - [T]^T[m_{Oe}^i]\{\ddot{r}_O\}_L \end{aligned} \tag{10}$$

or:

$$[M_e]\{\ddot{\Delta}_e\} + [C_e]\{\dot{\Delta}_e\} + ([K_e] + [K_e(\varepsilon)] + [K_e(\omega)])\{\Delta_e\} = \{F_e^c\} + \{F_e^d\} - \{F_e^i(\varepsilon)\} - \{F_e^i(\omega)\} - \{F_{Oe}^i\} \tag{11}$$

The notations are obvious. For all system Eq. (11) offers the set of equations:

$$\begin{bmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_n \end{bmatrix} \begin{Bmatrix} \ddot{\Delta}_1 \\ \ddot{\Delta}_2 \\ \vdots \\ \ddot{\Delta}_n \end{Bmatrix} + \begin{bmatrix} C_1 & & & 0 \\ & C_2 & & \\ & & \ddots & \\ 0 & & & C_n \end{bmatrix} \begin{Bmatrix} \dot{\Delta}_1 \\ \dot{\Delta}_2 \\ \vdots \\ \dot{\Delta}_n \end{Bmatrix}$$

$$
+\begin{bmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{bmatrix}\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{Bmatrix} = \begin{Bmatrix} \{F_e^c\} + \{F_e^d\} - \{F_e^i(\varepsilon)\} - \{F_e^i(\omega)\} - \{F_{Oe}^i\} \\ \{F_e^c\} + \{F_e^d\} - \{F_e^i(\varepsilon)\} - \{F_e^i(\omega)\} - \{F_{Oe}^i\} \\ \vdots \\ \{F_e^c\} + \{F_e^d\} - \{F_e^i(\varepsilon)\} - \{F_e^i(\omega)\} - \{F_{Oe}^i\} \end{Bmatrix} \tag{12}
$$

The finite elements are connected by the common nodes so that the number of degrees of freedom of the entire system is lower than that provided by the individual equations of motion of the finite elements. If the vector of independent coordinates is denoted by $\{\Delta\}$, it can be written:

$$
\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{Bmatrix} = [T]\{\Delta\} \ ; \quad \begin{Bmatrix} \dot{\Delta}_1 \\ \dot{\Delta}_2 \\ \vdots \\ \dot{\Delta}_n \end{Bmatrix} = [T]\{\dot{\Delta}\} \ ; \tag{13}
$$

$$
\begin{Bmatrix} \ddot{\Delta}_1 \\ \ddot{\Delta}_2 \\ \vdots \\ \ddot{\Delta}_n \end{Bmatrix} = [T]\{\ddot{\Delta}\} \tag{14}
$$

where $n$ is the total number of finite elements used.

$$
[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + ([K] + [K(\varepsilon)] + [K(\omega)])\{\Delta\} = \{F\} + \{F^*\} - \{F^i(\varepsilon)\} - \{F^i(\omega)\} - [F_O^i] \ . \tag{15}
$$

## 3. MODAL ANALYSIS

### 3.1. Basic notion

The equations that describe the evolution of an elastic mechanical system are [49]:

$$
[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{F\} \ , \tag{16}
$$

Considering also damping (manifested in the form of frictional forces proportional to the relative speeds), Eq.(16) becomes:

$$
[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{F\} \ . \tag{17}
$$

(similar to (20)) where matrices $[M]$, $[C]$, $[K]$ are symmetric, positive definite, and constants. Sometimes these matrices can be time-dependent. In these cases, an incremental study of the problem can be conducted in intervals, during which it is assumed that the matrix coefficients are constant.

The modal superposition method is not only a numerical procedure for solving differential equations with constant coefficients, but also an intuitive aid for understanding the problem from a physical point of view. The method consists of using a linear transformation of the form:

$$
\{\Delta\} = [\Phi]\{q\} \ , \tag{18}
$$

for bringing the system (21/22) to a simpler form. The matrix $[\Phi]$ is called the modal matrix. For equations (16), if the matrices are symmetric and positive definite, this transformation always exists, while for (17) $[C]$ must meet certain conditions, which will be presented in the study. Using (18), by a suitable choice of the modal matrix, it is possible for the system to break *into* a number of *n* second-order independent differential equations having constant coefficients, whose solution is no longer a problem.

Looking for a solution of the form:

$$
\{\Delta(t)\} = \{X\} \cos(\omega t + \psi) \ , \tag{19}
$$

for (21) it results in the condition:

$$
([K] - \omega^2[M])\{X\} \sin(\omega t + \psi) = 0 \ , \tag{20}
$$

*at any* moment of time $t$, which requires:

$$
([K] - \omega^2[M])\{X\} = 0 \ , \tag{21}
$$

To have for a linear, homogeneous system (21) other solutions besides the trivial one, zero, must have:

$$
det([K] - \omega^2[M]) = 0 \ . \tag{22}
$$

It is denoted by:

$$
P(\omega^2) = det([K] - \omega^2[M]) . \tag{23}
$$

the characteristic polynomial. The equation P(λ) = 0 is the characteristic equation. The values $\omega_i = \sqrt{\lambda_i} = 2\pi f_i$ represent the eigenpulsations of the system and $f_i$ the eigenfrequencies. The pulsation with the lowest value is called the fundamental pulsation.

The solution of Eq. (21), which uses the previously determined eigenvalue $\omega_i$ is denoted by $\{X_i\}$:

$$
([K] - \omega_i^2[M])\{X_i\} = 0 \ . \tag{24}
$$

The solution $\{X_i\}$ is the *i*- rank eigenvector of the system. The i-rank *mode of motion (eigenmode)* if a scaling mode (normalization) of its components (see [50],[51]). Denote the eigenmodes with $\{\Phi_i\} = \mu_i \{X_i\}$. Their ensemble forms the modal matrix $[\Phi]$:

$$[\Phi] = [\Phi_1 \; \Phi_2 \; \ldots \Phi_n] \; . \qquad (25)$$

The scalar $\mu$ depends on how the components of $\{X_i\}$ are normalized.

### 3.2. Properties of eigenvalues

**P1**. If the matrix $[K]$ is singular, then $\omega^2 = 0$ is an eigenvalue. The rigid motion mode is characterized by a zero eigenvalue.

**P2**. If $[K]$ and $[M]$ are symmetric, the eigenvalues are real.

**P3**. If $[K]$ and $[M]$ is positive definite, the eigenvalues are positive.

### 3.3. Properties of orthogonality

If i ≠ j the eigenmodes $\{\Phi_i\}$ have the properties:

$$\{\Phi_j\}^T [M] \{\Phi_i\} = 0 \qquad (26)$$

and:

$$\{\Phi_j\}^T [K] \{\Phi_i\} = 0 \; . \qquad (27)$$

The eigenmodes are said to be orthogonal by $[M]$ respectively by $[K]$.

If i=j, it is written:

$$m_i = \{\Phi_i\}^T [M] \{\Phi_i\}; \; k_i = \{\Phi_i\}^T [K] \{\Phi_i\} = 0 \; . \qquad (28)$$

**Generalized orthogonality relations [41]**

From the relationship:

$$[K]\{\Phi_j\} - \omega_j^2 [M]\{\Phi_j\} = 0 \; , \qquad (29)$$

can be expressed $\{\Phi_j\}$ in two ways:

i) $$\{\Phi_j\} = \frac{1}{\omega_j^2} [M]^{-1}[K]\{\Phi_j\} \qquad (30)$$

ii) $$\{\Phi_j\} = \omega_j^2 [K]^{-1}[M]\{\Phi_j\} = \omega_j^2 \left([M]^{-1}[K]\right)^{-1} \{\Phi_j\} \qquad (31)$$

relationship valid for $p = 0,1,2,3,\ldots$

Using the two relations, a single form can be obtained that represents the generalized orthogonality properties:

$$\{\Phi_i\}^T [M] \left([M]^{-1}[K]\right)^p \{\Phi_j\} = 0 \; , \; p \in Z \; . \qquad (32)$$

### 3.4. Definition relations for the damping matrix

The relation (32) offers a possibility to define the damping matrix in such a way that these orthogonality relations can be used. Thus, the damping matrix can be expressed as a linear combination of orthogonal matrices $[M]\left([M]^{-1}[K]\right)^r$ (Caughey representation):

$$[C] = \sum_{r \in Z} \alpha_r [M] \left([M]^{-1}[K]\right)^r \text{ with: } \alpha_r \in R \; . \qquad (33)$$

Considering a system with *n* independent coordinates, we keep only the linear combination with *n* independent terms:

$$[C] = \sum_{r=0}^{n-1} \alpha_r [M] \left([M]^{-1}[K]\right)^r = [M] \sum_{r=0}^{n-1} \alpha_r \left([M]^{-1}[K]\right)^r \qquad (34)$$

The Rayleigh damping matrix is:

$$[C] = [C_o] + [C_1] = \alpha_o [M] + \alpha_1 [K] \qquad (35)$$

### 3.5. Decoupling the equations of motion

The system (17) becomes, with the transformation (19), a system in $\{q\}$:

$$[M][\Phi]\{\ddot{q}\} + [C][\Phi]\{\dot{q}\} + [K][\Phi]\{q\} = \{F\} \qquad (36)$$

If $[C]$ has the form (34), the system (36) will be decoupled by multiplying by $[\Phi]^T$:

$$[\Phi]^T [M][\Phi]\{\ddot{q}\} + [\Phi]^T [C][\Phi]\{\dot{q}\} + [\Phi]^T [K][\Phi]\{q\} = [\Phi]^T \{F\} \qquad (37)$$

The matrices $[\Phi]^T [M][\Phi]$, $[\Phi]^T [C][\Phi]$ and $[\Phi]^T [K][\Phi]$ become diagonal. It is denoted:

$$[\backslash\, M^* \,\backslash] = [\Phi]^T [M][\Phi] \qquad (38)$$

$$[\backslash\, C^* \,\backslash] = [\Phi]^T [C][\Phi] \qquad (39)$$

$$[\backslash\, K^* \,\backslash] = [\Phi]^T [K][\Phi] \qquad (40)$$

In the following, the notations $m_i$, $k_i$ (see (28)), will be used for the diagonal components of the matrices (38),(40), and with:

$$c_i = [\Phi_i]^T [C][\Phi_i]; \qquad (41)$$

(if i ≠ j there are the relations $\{\Phi_j\}^T [C]\{\Phi_i\} = 0$)

In the hypothesis that the eigenvalues are non-zero and distinct, the system (16)/(17) is decoupled into *n* second-order independent differential equations, with constant coefficients.

## 4. APPLICATION OF MODAL ANALYSIS TO FLEXIBLE MBS SYSTEMS

Considering an MBS, the evolution of the flexible mechanical system is described by Eq. (15):

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + ([K] + [K(\varepsilon)] + [K(\omega)])\{\Delta\} = \{F_{tot}\} \qquad (42)$$

in which the notation was used:

$$\{F_{tot}\} = \{F\} + \{F^*\} - \{F^i(\varepsilon)\} - \{F^i(\omega)\} - [F_O^i] \quad (43)$$

for the total generalized force. If the system is considered unexcited, the homogeneous differential system becomes:

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + ([K] + [K(\varepsilon)] + [K(\omega)])\{\Delta\} = 0 \quad (44)$$

where $[M],[K]$, $[K(\omega)]$ are symmetric matrices, $[C]$ and $[K(\varepsilon)]$ are skew-symmetric matrices.

Consider now the reduced form equation:

$$[M]\{\ddot{\Delta}\} + ([K] + [K(\omega)])\{\Delta\} = 0 \quad . \quad (45)$$

The polynomial:

$$det([K] + [K(\omega^2)] - \omega^2[M]) = 0 \quad , \quad (46)$$

offers us the eigenvalues and the modal matrix $[\Phi]$ for the system (50).

If we consider the transformation (18), introducing $\{\Delta\}$ in Eq. (42) and pre-multiplying the system with $[\Phi]^T$ it obtains:

$$[\Phi]^T[M][\Phi]\{\ddot{q}\} + [\Phi]^T[C][\Phi]\{\dot{q}\} + [\Phi]^T[K][\Phi]\{q\} + [\Phi]^T[K(\varepsilon)][\Phi]\{q\} + [\Phi]^T[K(\omega)][\Phi]\{q\} = [\Phi]^T\{F\} \quad (47)$$

If $[C]$and $[K(\varepsilon)]$ are skew-symmetric matrices it obtains:

$$[\Phi]^T[C][\Phi] = 0\,;[\Phi]^T[K(\varepsilon)][\Phi] = 0\,; \quad (48)$$

and the system (47) becomes:

$$[\Phi]^T[M][\Phi]\{\ddot{q}\} + [\Phi]^T[K][\Phi]\{q\} + [\Phi]^T[K(\omega)][\Phi]\{q\} = [\Phi]^T\{F\} \quad (49)$$

in terms of modal coordinates. So, in this way, solving the system (47) returns to solving the system (49) in modal coordinates:

$$m_1\ddot{q}_1 + k_1q_1 = F_1$$
$$m_2\ddot{q}_2 + k_2q_2 = F_2$$
$$\ldots.$$
$$m_n\ddot{q}_n + k_nq_n = F_n \quad (50)$$

If the viscous damping is also considered in the motion equations, all the reasoning made remains valid; equations (50) will contain an additional term, determined by the viscous damping $c_i$ defined in (41).

The tubular chassis of a car (Figure 1) for the Formula Student race is made up of tubular bars that form a structure with several objectives. The shape of this structure, a tubular chassis, was determined by years of experience. After several hundred racing vehicle models, decades of research and development led to the current version and chassis manufacturing process. The focus of the current investigation is the chassis's vibration characteristics. The pilots' observations about the presence of some unwanted vibrations, particularly when navigating curves, led to the research's conclusion that the chassis needed to be stiffened.

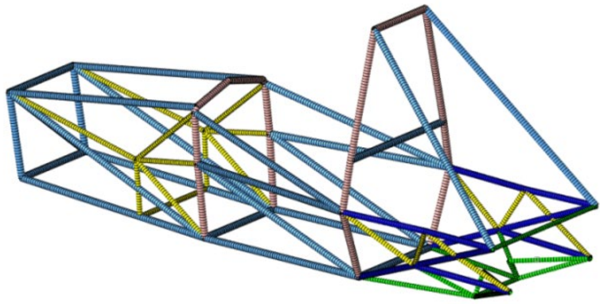

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**Figure 1.** The tubular chassis

For modeling and calculations, the Altair HyperWorks package and FEM are used. Both the original and revised versions of the structure are subjected to the calculus. The rear portion of the structure has been modified to improve the chassis's behavior. Fig. 1 shows the structure in its original form. A realistic simulation environment is offered by FEM. The performances of the proposed structural solution can therefore be easily displayed.

More bars were added to the framework to boost its torsional stiffness.

**Table 1.** The eigenfrequency [Hz] -

| *Mode* | *Frequency [Hz]* |
|---|---|
| 1 | 41.9 |
| 2 | 45.6 |
| 3 | 66.9 |
| 4 | 73.3 |
| 5 | 90.9 |
| 6 | 102.6 |
| 7 | 103.4 |
| 8 | 118.5 |
| 9 | 125.7 |
| 10 | 129.2 |
| 11 | 130.3 |
| 12 | 133.4 |
| 13 | 139.1 |
| 14 | 149.3 |

Table 1 shows the natural frequencies obtained from the finite element analysis. The tables did not include the six natural pulsations that equaled zero, corresponding to rigid-body motion. Figures 2-7 show the structure's eigenmodes of vibration.

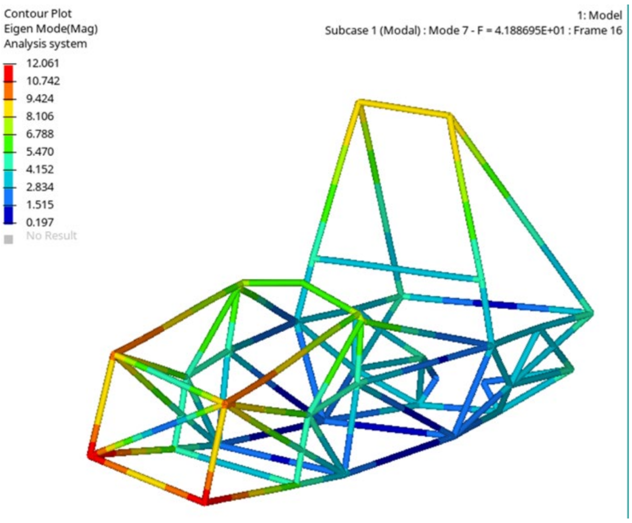


**Figure 2.** The 1st eigenmode

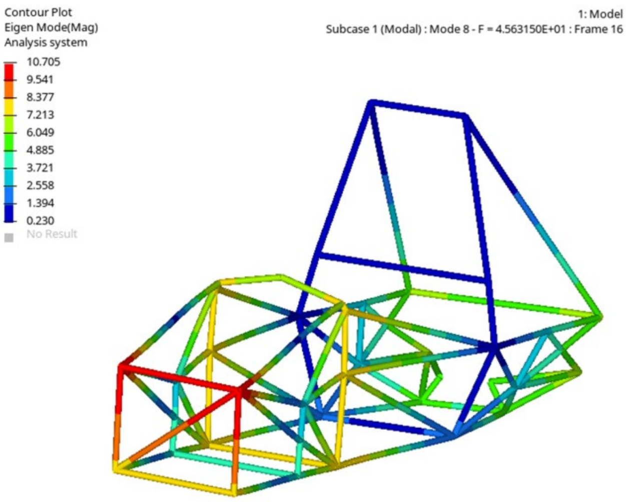


**Figure 3.** The 2nd eigenmode

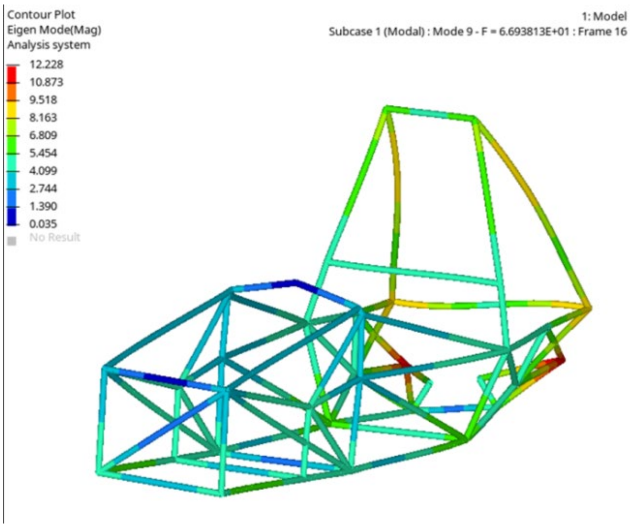


**Figure 4** The 3rd eigenmode

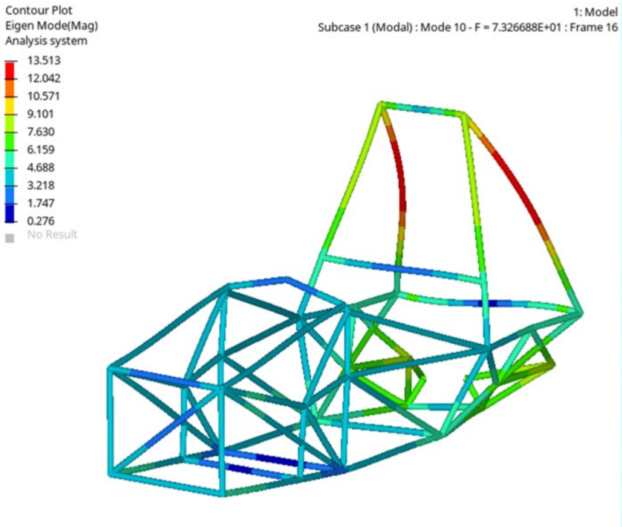


**Figure 5.** The 4th eigenmode

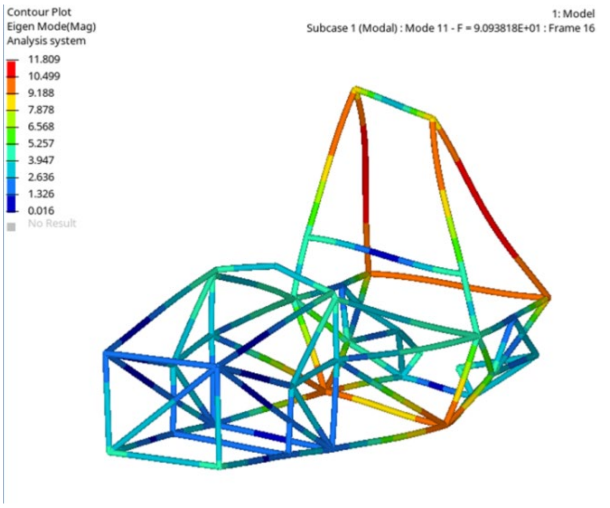


**Figure 6.** The 5th eigenmode

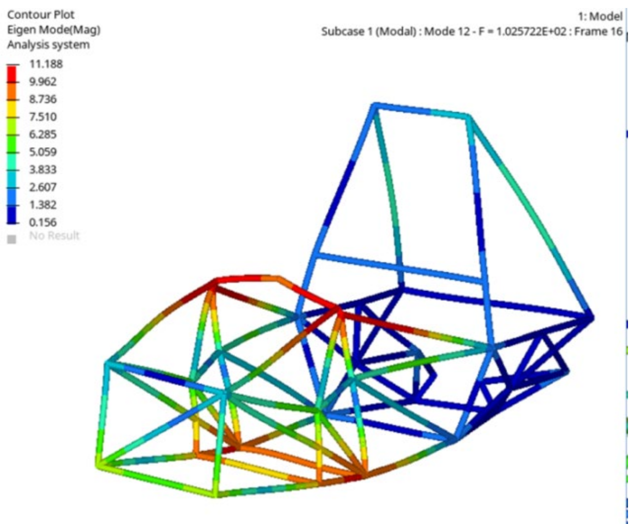


**Figure 7.** The 6th eigenmode

## 5. CONCLUSIONS

In the case of the motion equations obtained for a flexible MBS, it is not possible to directly apply the modal analysis method, primarily due to an additional term that appears in these equations, represented by a skew-symmetric matrix. Likewise, the stiffness matrix loses symmetry when a stiffness term represented by a skew-symmetric matrix is incorporated. However, the orthogonality properties of classical modal analysis also apply in this case and can be useful for analysis in certain contexts, as detailed in the paper. The skew-symmetric matrix that appears in the equations as a proportional damping matrix is actually a "conservative" matrix, that is, its existence does not cause energy to dissipate in the system through friction. In conclusion, modal analysis, despite its current limitations in this case, can also be a useful method for obtaining the dynamic response of MBS systems.

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