Use of the Whittaker Equations in FEA of Two-Dimensional Elastic Multibody Systems

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Abstract: - In a previous paper [1], methods for obtaining the equations of motion for an MBS system with elastic elements were presented, using the finite element method (FEM), for the case of planar systems, composed of elastic plates, where two-dimensional finite elements were used. In the present paper, the methods presented are extended for the case of scleronomic, holonomic systems, using Whittaker's equations. The results contribute to the development of the study of the field of multibody systems with elastic elements. This development is imposed by the increase in the performance of machines and mechanisms, which has led to an increase in the operating speeds and forces developed in these systems. The result was that the elasticity of the various elements of a mechanical system can significantly influence its behavior, and undesirable phenomena such as vibrations or loss of stability may appear. In the paper, a classical method from Analytical Mechanics is applied, in parallel with FEM, for the study of mechanical systems with certain particularities in plane motion. The main advantage offered by this approach is the reduction of the number of independent coordinates necessary to describe the motion of a finite element, ultimately resulting in the reduction of the number of differential equations describing the motion of the entire system.

Keywords: - vibration, two-dimensional FE, Lagrange's equations, Whittaker equations, holonomic liaisons, MBS, FEM.

1. INTRODUCTION

The development of the industry in the last years has led to the need to know the behavior of multibody systems (MBS), which currently operate at high speeds and with increased forces. An aspect that must be taken into consideration is the multitude of parameters necessary to describe these systems, which implies the automation of calculations to consider a greater number of cases. Obviously, with increasing operating speeds and forces that occur, the behavior of MBS systems will be influenced by the elasticity of the component elements, which can no longer be neglected or ignored. The analysis of these MBS is done using Analytical Mechanics in parallel with the Finite Element Method (FEM). There are a number of works that deal with the fundamentals of this analysis.

To obtain the dynamics of the system, a number of studies in the fields of car modeling, structural mechanics, engineering structures dynamics, microelectromechanical systems (MEMS), and so on require the use of both MBS and FEM. Reducing the dimension of the FEM model and then importing it into an MBS code for additional simulation is how the FEM-MBS coupling is achieved in [2]. The theoretical foundation of the FEM-MBS interface (MORPACK) is provided in this article. It allows the use of any reduction technique for FE-modeled structures and additionally imports them (Ritz approximation) into SIMPACK through the creation of SID files. Press systems with ever higher stroke rates are being used in contemporary automobile press shops to meet energy and efficiency targets. The dynamic loads and particularly on the forming tool, rise as a result [3]. The dynamic behavior of the blank holder, the heaviest moving part of the shaping tool, is the primary emphasis of the study. A linked MBS-FEM simulation, which blends stiff and elastic modelling techniques, is used to forecast those vibrations. Additionally, the vibration of the blank holder under operating load is experimentally validated. The MBS-FEM simulation and the test measurements correlate well. A thorough dynamic model of the train-track system is created using FEM (NASTRAN) and the commercial MBS analysis software SIMPACK. This model is used to analyze and compare the behavior of a ballasted track in comparison with three types of weak track [4]. Consequently, the MBS program incorporates a combined MBS-FEM representation, enabling the investigation of loads throughout a broad frequency range. Some conclusions about the potential validity of very simple train-track interaction model types are presented after an analysis of the quality of the data produced with the various model types used. To adjust the FEM model, a model-updating technique is suggested [5]. The surrogate model (having a multi-objective function) that takes into account the amplitude and form of each measured curve is suggested as a solution to the complexity of many measured responses. The FEM is then calibrated using the new results once the surrogate model parameters have been optimized. The kinematics description of the elastic body in MBS, using the Rayleigh-Ritz method, and d'Alembert motion equations and Jourdain's virtual power principle, is all thoroughly examined in [6] by merging the theories of multi-body systems (MBS) and FE. Additionally examined and evaluated are the processing techniques used to apply the information in the vehicle system. According to the aforementioned ideas, the superelement approach significantly lowers the DOF of the frame stated in MBS by condensing the entire FE model.

The particular MBS framework on which the approach is applied—the formulation of motion equations being the first step—is one particular factor that must be taken into account while using FEM for dynamic analysis. At this point, the motion laws start to include new terms that reflect the effects of the various types of acceleration. Choosing the appropriate approximations to create equations of motion that as faithfully represent reality as possible is the difficulty that arises in modeling.

The specificity in creating the equations is represented by these new terms. Analytical mechanics is one method of obtaining the equations of motion [7-9], as its methods offer the greatest degree of generality when dealing with an MBS problem. Furthermore, while examining modern systems, a wide range of stiff or elastic components with various properties must be taken into account. The systems being studied are by no means simple and necessitate systematic design and numerical modeling efforts.

These scenarios make it obvious that any advantage a chosen method may offer is significant and ought to be taken into account for applications in terms of modeling, the volume of computations needed, and simulation costs. The final step is to derive the equations of motion in order to create user-friendly software and efficient algorithms. Analytical mechanics theory allows for the deployment of extremely generic approaches. This makes it possible to handle different applications consistently. The methods that analytical mechanics offers are crucial to the analysis [10-15]. This method can also be used to express the fundamental laws of mechanics in a number of related ways. Depending on their experience and the particular issue being studied, researchers employ a variety of methodologies, taking into account the benefits and limitations of each.

Nowadays, the majority of researchers employ Lagrange's equations as standard methodology, primarily because it ensures a reasonable degree of generality for many issues encountered in industrial applications and makes use of relatively simple mechanical concepts that researchers are familiar with (energy, work, or momentum). Consequently, it is possible to precisely describe the potential constraints. The application of Lagrange's equations for the modeling and study of the dynamic behavior of mechanical systems is covered in a large number of references in the literature. Articles [16–19] are highlighted in this work. Analytical mechanics allows for the freedom to choose the method for deriving the equations of motion, which can be very advantageous in dynamic investigations for particular engineering applications and circumstances.

These advantages, which have reduced analysis times and made modeling easier, have been recognized and utilized by certain researchers [20–22]. It is clear that the employment of analogous methodologies also spurred research into the possibility of using the best numerical analysis software in order to ensure an optimal approach to the analysis. Of course, a crucial element of these methods is the FEM [23–25].

Analytical mechanics provides equivalent concepts, but applications of these formulas require appropriate numerical techniques. These methods, which help reduce the costs of numerical analysis, are explained in [26]. Symbolic formalism is a helpful method for cutting down on the amount of time needed to model such systems [27].

The most important sizes of an MBS's overall rigid movement are elastic body deformations. A systematic effort to facilitate the writing of algorithms in MBSs is presented in [28]. Numerous fields make use of MBS approaches [29]. Models for tackling MBS problems in a variety of fields are offered in [30-32]. In addition to studying these systems, some technical applications are examined. Current industrial applications need the study of complex systems, and modeling and simulating such systems necessitate significant financial outlays for infrastructure (hardware and software) and human resources. Thus, the goal is to produce the most accurate findings while achieving the simplest possible trade-off across modeling systems.

Numerous techniques for symbolically expressing motion equations and cutting down on modeling time have been developed throughout the last ten years. Two methods that offer useful models for a specific elastic MBS are examined in [33, 34]. [35,36] outline the traditional methods employed in the MBS. With differing degrees of effectiveness, a great deal of research has been done on this topic [37]. The conventional approaches of tackling these issues are explained in [38], with different solutions for different uses [39]. The usage of a composite material in the production of an MBS is investigated in [40]. The purpose of articles [41-43] is to write equations of motion to explore a generic MBS.

Since many engineering systems are planar, two-dimensional elements are frequently used in practical applications [44]. By looking at the natural frequencies for system optimization, [45] gives an example of how to apply two-dimensional thin plates in a real-world issue. The modeling technique used for a multibody system was the transfer matrix method. Together with a few particular examples that bolstered the study, the theoretical foundations of this methodology were also covered. In [46], the wing skin of an airplane is studied by a bidimensional finite element analysis. The six degrees of freedom for rigid body motion in the examined plane were superimposed with vibrations.

Decisions were made during the plane's design phase based on the study, which examined the impact of elasticity on total motion. A shell element with rotating blade geometrical properties was developed in [47]. By contrasting its predictions with other study findings from the literature, the proposed model was verified. FEM was used in [48] to create a model for huge structures. The advantages of the suggested approach were illustrated through the synthesis of a spacecraft's dynamic model. The ship had two flexible and adjustable solar panels. The use of two-dimensional shell finite elements in an FE model was investigated. A new kind of finite element with a thin hyper-elastic shell was presented in [49].

The concept was founded on the Kirchhoff–Love theory. Other interesting results are reported in [50–59]. A detailed examination of these issues is required because the use of models for planar structures, or MBS, in the industrial sector is fraught with challenges. This study highlights models used for analyzing planar structures or MBS with elastic elements using the finite element method, more specifically, the two-dimensional finite element model, taking into consideration its planar motion, by closely examining the main methods offered by analytical mechanics in its classic form.

One of the methods that allows reducing the size of the analyzed problems is given by Whittaker [60]. Being a useful method, the developments of this theory and examples have been analyzed by various researchers [61-64]. The present work aims to apply this method to the study of MBS systems with elastic elements when FEM is used.

2. MOTION AND MODEL

2.1. Kinematics

The challenge in dynamically analyzing MBS with elastic components is to employ suitable models that faithfully represent the variables affecting the system's response. Engineers and researchers now recognize FEM as a technique for solving these kinds of problems. It suggests that the system's evolution throughout time can be ascertained by applying the basic laws of mechanics. Although a variety of techniques are employed to do this, the Method of Lagrange's Equations (LE) is still the most widely used approach. Recent practical applications involving large-scale, complex systems have led to the conclusion that other analytical mechanics-based techniques may be more beneficial to obtain the evolution of motion.

As a result, this paragraph methodically outlines a few approaches that can be employed in place of the LE Method and chosen for use in particular problem-solving situations where their benefits are clear.

A specific two-dimensional finite element is taken into consideration and described in order to perform this analysis [65-67].

The unit vectors $\bar{i}$ and $\bar{j}$ are associated with the finite element, which is associated with a local, mobile reference frame *Oxy*. The velocity of the origin of the local coordinate system is $\bar{v}_o(v_{O1}, v_{O2}j)$ and $\bar{a}_o(a_{O1}, a_{O2}j)$ is the acceleration, $\bar{\omega} = \omega\bar{k}$ is the angular velocity and $\bar{\varepsilon} = \varepsilon\,\bar{k}$ the angular acceleration. It is assumed that the elements' deformation occurs in the motion plane.

In the following, the index L is employed to specify the local (mobile) coordinate system having the origin in *O* (and $\bar{r}_O$ is it the position vector), and *G* specifies the global (fixed) coordinate system with the origin in O_1. The rotation angle of the local system relative to the global one is *θ*. Knowing the components of a vector in *L*, these can be determined in *G* using the transformation:

$$\{v\}_{Ox_1y_1} = [R]\{v\}_{Oxy} \tag{1}$$

where the rotation matrix has the well known form:

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

In the paper, the following notations will be used:

- $(X_{O,1}, X_{O,2})$ are the components of $\bar{r}_O$ in *G* frame;
- $(x_{O,1}, x_{O,2})$ are the components of $\bar{r}_O$ in the *L* frame;
- $(X_{M,1}, X_{M,2})$ is used for the coordinates of the position vector $\bar{r}_M$ expressed in the *G* frame;
- $(x_{M',1}, x_{M',2})$ for the same vector in the *L* frame.
- $(X_{M',1}, X_{M',2})$ is used for the coordinates of the position vector $\bar{r}_{M'}$ of the point *M’* (*M* after displacement/deformation) in the *G*;
- $(x_{M',1}, x_{M',2})$ the same expressed in L;
- (X_1, X_2) is used for the coordinates of the position vector $\bar{r}$ of point *M* with respect to origin *O*, in *G* ;
- (x_1, x_2), for the same vector expressed in *L*;
- $\bar{u} = \overline{MM'}$, is the displacement vector with the components (u, v).

After the deformation of an arbitrary point *M* of an elementary element becomes *M’*. The coordinates of *M’* in a fixed (G) coordinate system are:

$$\begin{aligned} X_{M',1} &= X_{O,1} + (x_1 + u)\cos\theta - (x_2 + v)\sin\theta; \\ X_{M',2} &= X_{O,2} + (x_1 + u)\sin\theta + (x_2 + v)\cos\theta. \end{aligned} \tag{3}$$

In FEM as the displacements are approximated by:

$$u = N_{1r}\delta_r \quad ; \quad v = N_{2r}\delta_r \quad ; \quad r = \overline{1,p}. \tag{4}$$

2.2. Basic concepts

2.2.1. Kinetic Energy

Considering a single finite element, the energy has the expression [39]:

$$E_C = \frac{1}{2}\int_V \rho v_M^2 dV = = \frac{1}{2}\int_V \rho\left[\left(\dot{x}_{M',1}\right)^2 + \left(\dot{x}_{M',2}\right)^2\right]dV \tag{5}$$

The following notations were introduced:

$$m = \int_V \rho dV\ ;\ J_O = \int_V \rho\left(x_1^2 + x_2^2\right)dV\ ; \tag{6}$$

$$m_{rt} = \int_V \rho\left(N_{1r}N_{1t} + N_{2r}N_{2t}\right)dV\ ; \tag{7}$$

$$S_1 = \int_V \rho x_1 dV \quad ; \quad S_2 = \int_V \rho x_2 dV \quad ; \tag{8}$$

$$m_{O,kr}^I = \int_V \rho N_{kr} dV \quad ; \quad m_{1,mr} = \int_V \rho\, x_1 N_{mr} dV\ ; \tag{9}$$

$$m_{2,mr} = \int_V \rho\, x_2 N_{mr} dV \quad , \quad k, m = 1{,}2\ ; \quad r = \overline{1,p}. \tag{10}$$

Considering the expression obtained before for $\dot{x}_{M',1}$ and $\dot{x}_{M',2}$ this energy can be expreed as: [21]:

$$\begin{aligned} E_c &= \frac{1}{2}m\left(\dot{x}_{O,1}^2 + \dot{x}_{O,1}^2\right) + \frac{1}{2}\omega^2 J_O \\ &+ \frac{1}{2}\omega^2\delta_t\delta_r m_{rt} + \frac{1}{2}\dot{\delta}_r\dot{\delta}_t m_{rt} - \omega\left(\dot{x}_{O,1}S_2 - \dot{x}_{O,2}S_2\right) \\ &- \omega\delta_r\left(\dot{x}_{O,1}m_{O,2r}^I + \dot{x}_{O,2}m_{O,1r}^I\right) + \omega^2\delta_r\left(m_{2,2r} + m_{1,1r}\right) \\ &+ \left(\dot{x}_{O,1}m_{O,1r}^I + \dot{x}_{O,2}\,\dot{\delta}_r m_{O,2r}^I\right)\dot{\delta}_r - \omega\dot{\delta}_r\left(m_{2,1r} - m_{1,2r}\right) \\ &- \omega\delta_r\dot{\delta}_t\left(m_{12,rt} - m_{21,rt}\right)\ . \end{aligned} \tag{11}$$

The material density ρ is considered as constant.

2.2.2. Potential Energy

The potential energy is [21,39]:

$$E_p = \frac{1}{2}\int_V \left(\sigma_{11}\varepsilon_{11} + 2\sigma_{12}\varepsilon_{12} + \sigma_{22}\varepsilon_{22}\right)dV \tag{12}$$

Here, there are used the relations:

$$\sigma_{11} = \frac{E}{1-2\mu}\varepsilon_{11} \quad ; \quad \sigma_{22} = \frac{E}{1-2\mu}\varepsilon_{22} \quad ; \quad \sigma_{12} = \frac{E}{2(1-\mu)}\varepsilon_{12} \tag{13}$$

The strains can be calculated using the definition relations:

$$\varepsilon_{11}=\frac{\partial u}{\partial x_1} \quad ; \quad \varepsilon_{22}=\frac{\partial v}{\partial x_2} \quad ; \quad \varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u}{\partial x_2}+\frac{\partial v}{\partial x_1}\right) \tag{14}$$

using the relation of the u and v expressed by (4), it obtained:

$$\varepsilon_{11}=\frac{\partial u}{\partial x_1}=\frac{\partial N_{1r}}{\partial x_1}\delta_r ;\ \varepsilon_{22}=\frac{\partial v}{\partial x_2}=\frac{\partial N_{2r}}{\partial x_2}\delta_r ; \quad \varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u}{\partial x_2}+\frac{\partial v}{\partial x_1}\right)=\frac{1}{2}\left(\frac{\partial N_{1r}}{\partial x_2}+\frac{\partial N_{2r}}{\partial x_1}\right)\delta_r \tag{15}$$

It is possible to obtain the stresses:

$$\sigma_{11}=\frac{E}{1-2\mu}\frac{\partial N_{1r}}{\partial x_1}\delta_r ; \sigma_{22}=\frac{E}{1-2\mu}\frac{\partial N_{2r}}{\partial x_2}\delta_r ; \quad \sigma_{12}=\frac{E}{4(1-\mu)}\left(\frac{\partial N_{1r}}{\partial x_2}+\frac{\partial N_{2r}}{\partial x_1}\right)\delta_r \ . \tag{16}$$

So, the potential energy becomes:

$$E_p=\frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}dV=\frac{1}{2}k_{rt}\delta_r\delta_t \ , \tag{17}$$

where the stiffness matrix is defined by [39]:

$$k_{rt}=\frac{1}{2}\int_V\left[\frac{E}{1-2\mu}\frac{\partial N_{1r}}{\partial x_1}\frac{\partial N_{1t}}{\partial x_1}+\frac{E}{1-2\mu}\frac{\partial N_{2r}}{\partial x_2}\frac{\partial N_{2t}}{\partial x_2}\right] +\frac{1}{2}\int_V\left[\frac{E}{8(1-\mu)}\left(\frac{\partial N_{1r}}{\partial x_2}+\frac{\partial N_{2r}}{\partial x_1}\right)\left(\frac{\partial N_{1t}}{\partial x_2}+\frac{\partial N_{2t}}{\partial x_1}\right)\right] \tag{18}$$

2.2.3. Work

It is considered the work of the generalized concentrated forces $q_i \ \ i=\overline{1,p}$ [21,29]:

$$W^c=q_i\delta_i \quad ; \quad i=\overline{1,p} , \tag{19}$$

and of the generalized volume forces $q_i^* \ i=\overline{1,p}$:

$$W^{\mathrm{d}}=\sum_{r=1}^{p} q_r^*\delta_r \quad . \tag{20}$$

Finally is obtained the total work, necessary for future developments:

$$W=\left(W^c+W^{\mathrm{d}}\right)=\left(q_i+q_i^*\right)\delta_i \quad ; \quad i=\overline{1,p} \ . \tag{21}$$

2.2.4. Lagrangian

The expression for the Lagrangian is:

$$L=E_c-E_p+W \ . \tag{22}$$

Eqs. (11), (17), (21) introduced in (22) offer the Lagrangian [65-67]:

$$\begin{aligned} L=&\frac{1}{2}m\left(\dot{x}_{O,1}^2+\dot{x}_{O,2}^2\right)+\frac{1}{2}\omega^2 J_O \\ &+\frac{1}{2}\omega^2\delta_t\delta_r m_{rt}+\frac{1}{2}\dot{\delta}_r\dot{\delta}_t m_{rt} \\ &-\omega\left(\dot{x}_{O,1}S_2-\dot{x}_{O,2}S_1\right)-\omega\delta_r\left(\dot{x}_O m_{O,2r}^I+\dot{x}_{O,2}m_{O,1r}^I\right) \\ &+\left(\dot{x}_{O,1}m_{O,1r}^I+\dot{x}_{O,2}\,\delta_r m_{O,2r}^I\right)\dot{\delta}_r+\omega^2\delta_r\left(m_{2,2r}+m_{1,1r}\right) \\ &-\omega\dot{\delta}_r\left(m_{2,1r}-m_{1,2r}\right)-\omega\delta_r\dot{\delta}_t\left(m_{12,rt}-m_{21,rt}\right) \\ &-k_{rt}\delta_r\delta_t+q_r\delta_r+q_r^*\delta_r \quad ; \quad r,t=\overline{1,p} \ . \end{aligned} \tag{23}$$

2.2.5. Hamiltonian

The expression of the Hamiltonian is [46,47]:

$$H=\sum_{r=1}^{p}\frac{\partial L}{\partial\dot{\delta}_r}\dot{\delta}_r-L \quad . \tag{24}$$

3. LAGRANGIAN FORMALISM

The differential equations of second order for any mechanical system can be determined using a number of comparable formalisms provided by analytical mechanics. This section outlines the primary formalisms used in the MBS with elastic elements that employ the Lagrange equations technique in conjunction with FEM.

The use of Lagrange's equations was helpful for dealing with problems similar to those in an MBS. It was advantageous to use scalars instead of vectors. Using Lagrange's equations, the generalized forces were calculated using the kinetic and potential energy formulas [1]:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\delta}_i}\right)-\frac{\partial L}{\partial\delta_i}=0 \quad ; \quad i=\overline{1,p} \tag{25}$$

$$\frac{\partial L}{\partial\dot{\delta}_r}=\dot{\delta}_t m_{rt}+\left(\dot{x}_{O,1}m_{O,1r}^I+\dot{x}_{O,2}m_{O,2r}^I\right) -\omega\left(m_{2,1r}-m_{1,2r}\right)-\omega\delta_r\left(m_{12,rt}-m_{21,rt}\right); r,t=\overline{1,p} \tag{26}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\delta}_r} = \ddot{\delta}_t m_{rt} + \left(\ddot{x}_{O,1} m^I_{O,1r} + \ddot{x}_{O,2} m^I_{O,2r}\right) - \varepsilon\left(m_{2,1r} - m_{1,2r}\right) - \varepsilon\delta_r\left(m_{12,rt} - m_{21,rt}\right) - \omega\dot{\delta}_r\left(m_{12,rt} - m_{21,rt}\right) \quad ; \quad r,t = \overline{1,p} \tag{27}$$

$$\frac{\partial L}{\partial \delta_r} = \omega^2 \delta_t m_{rt} - \omega\left(\dot{x}_{O,1} m^I_{O,2r} + \dot{x}_{O,2} m^I_{O,1r}\right) + \omega^2\left(m_{2,2r} + m_{1,1r}\right) - \omega\dot{\delta}_t\left(m_{12,rt} - m_{21,rt}\right) - k_{rt}\delta_t + q_r + q^*_r \quad ; \quad r,t = \overline{1,p} \tag{28}$$

Using Eq. (27) and Eq. (28) in Eq. (25) it results:

$$m_{rt}\ddot{\delta}_t + 2\omega\dot{\delta}_r\left(m_{12,rt} - m_{21,rt}\right) + \left[k_{rt} - \varepsilon\left(m_{12,rt} - m_{21,rt}\right) - \omega^2\delta_t m_{rt}\right]\delta_t = -\left(\ddot{x}_{O,1} m^I_{O,1r} + \ddot{x}_{O,2} m^I_{O,2r}\right) + \varepsilon\left(m_{2,1r} - m_{1,2r}\right) - \omega\left(\dot{x}_{O,1} m^I_{O,2r} + \dot{x}_{O,2} m^I_{O,1r}\right) + \omega^2\left(m_{2,2r} + m_{1,1r}\right) + q_r + q^*_r \quad ; \quad r,t = \overline{1,p} \tag{29}$$

4. WHITTAKER FORMALISM

We examine the scenario of a conservative mechanical system with holonomic liaisons, where the Lagrange function solely depends on the generalized coordinates and velocities (does not depend explicitly on time). As a result, the Lagrange function will have the following form:

$$L = L(q_1, q_2, \ldots \; ..q_p, \dot{q}_1, \dot{q}_2, \ldots \; ..\dot{q}_p,) \quad . \tag{30}$$

Any of the traditional techniques can be used to study the motion of such a system.

It can be demonstrated that for such a system (scleronomic and conservative), the number of differential equations found can be decreased by one, providing *n-1* differential equations of second order that are enough to define the motion. [60].

The Jacobi energy integral, which is appropriate for conservative scleronomic systems, will be used in the following.

$$\sum_{i=1}^{p} \frac{\partial L}{\partial \dot{\delta}_i}\dot{\delta}_i - L = H = ct \quad . \tag{31}$$

Using the notations:

$\frac{dq_i}{dt} = \dot{q}_i$ and making the convention that in the case of the derivative with respect to the variable τ the notation is used: $\frac{dq_i}{d\tau} = q'_i$, the Lagrange function becomes:

$$L = L(\tau, q_2, \ldots \; ..q_p, \dot{q}_1, \dot{q}_2, \ldots \; ..\dot{q}_p,) \quad . \tag{32}$$

In Eq.(32) and in the following, in the Lagrange function, one of the *n* generalized coordinates, namely q_1, is denoted by τ :

$$q_1 = \tau \, . \tag{33}$$

Whittaker demonstrated that a system of *n* Lagrange equations may be converted into a system of *n-1* equations with forms identical to Lagrange's equations [60]:

$$\frac{d}{d\tau}\left(\frac{\partial L^*}{\partial \delta'_i}\right) - \frac{\partial L^*}{\partial \delta_i} = 0 \quad ; \quad i = \overline{2,p} \tag{34}$$

with a different function L^*, known as the Whittaker function, used in place of the Lagrange function. One of the generalized coordinates is taken into consideration as an independent variable instead of time *t*. This will be indicated in the following by τ .

In such an approach, the generalized velocities (time derivatives of the generalized coordinates) have the form:

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{dq_i}{dq_1}\cdot\frac{dq_1}{dt} = = q'_i\,\dot{\tau} = \begin{cases} \dot{\tau} & i = 1 \\ q'_i\,\dot{\tau} & i \geq 2 \end{cases} , \; i = \overline{1,n} \tag{35}$$

It can be written too:

$$q'_i = \frac{1}{\dot{\tau}}\dot{q}_i \quad , \quad i = \overline{1,n} \tag{36}$$

From here, you can also get:

$$\frac{\partial q'_i}{\partial \dot{q}_i} = \frac{1}{\dot{\tau}} \quad , \quad i = \overline{1,n} \tag{37}$$

After substituting Eq.(35),(36) into the Lagrange function, a new function equal to the Lagrange function is obtained,

$$L = L(\tau, q_2, \ldots \; ..q_p, \dot{\tau}, q'_2\,\dot{\tau}, \ldots \; ..q'_p\,\dot{\tau}) = \Omega(\tau, q_2, \ldots \; ..q_p, \dot{\tau}, q'_2, \ldots \; ..q'_p) \quad . \tag{38}$$

In the following, it will be calculated the partial derivatives of the L function and of the function in both the case of generalized coordinates and generalized velocities,

$$\frac{\partial L}{\partial \delta_i} = \frac{\partial \Omega}{\partial \delta_i} \quad ; \quad i = \overline{1,p} \quad ; \tag{39}$$

$$\frac{\partial L}{\partial \dot{\delta}_1} = \frac{\partial L}{\partial \dot{\tau}} = \frac{\partial \Omega}{\partial \dot{\tau}} + \sum_{j=2}^{p}\frac{\partial \Omega}{\partial q'_j}\frac{\partial q'_j}{\partial \dot{\tau}} = = \frac{\partial \Omega}{\partial \dot{\tau}} - \sum_{j=2}^{p}\frac{\partial \Omega}{\partial q'_j}\frac{1}{\dot{\tau}^2}\dot{q}_j, \quad i = 1 \tag{40}$$

$$\frac{\partial L}{\partial \dot{\delta}_i}=\frac{\partial \Omega}{\partial q'_i}\frac{\partial q'_i}{\partial \dot{q}_i}=\frac{1}{\dot{\tau}}\frac{\partial \Omega}{\partial q'_i}, \quad i=\overline{2,p} \tag{41}$$

From Eq.(40) it obtained:

$$\frac{\partial \Omega}{\partial \dot{\tau}}=\frac{\partial L}{\partial \dot{\delta}_1}+\sum_{j=2}^{p}\frac{\partial \Omega}{\partial q'_j}\frac{1}{\dot{\tau}^2}\dot{q}_j \tag{42}$$

and from Eq.(41):

$$\frac{\partial \Omega}{\partial q'_i}=\dot{\tau}\frac{\partial L}{\partial \dot{\delta}_i}, \quad i=\overline{2,p} \tag{43}$$

Introducing (43) into (42) it obtained too:

$$\frac{\partial \Omega}{\partial \dot{\tau}}=\frac{\partial L}{\partial \dot{\delta}_1}+\sum_{j=2}^{p}\frac{\partial L}{\partial \dot{q}_j}\frac{1}{\dot{\tau}}\dot{q}_j \tag{44}$$

or:

$$\dot{\tau}\frac{\partial \Omega}{\partial \dot{\tau}}=\frac{\partial L}{\partial \dot{\delta}_1}\dot{\tau}+\sum_{j=2}^{p}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j$$

$$=\frac{\partial L}{\partial \dot{\delta}_1}\dot{\delta}_1+\sum_{j=2}^{p}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j=\sum_{j=1}^{p}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j \tag{45}$$

It can be seen that the right-hand side of the previous equality occurs in the energy integral Eq.(31). This gives another expression for Eq.(31):

$$\dot{\tau}\frac{\partial \Omega}{\partial \dot{\tau}}-L=H=ct \ . \tag{46}$$

Because $L=\Omega$, Eq.(46) can be written:

$$\dot{\tau}\frac{\partial \Omega}{\partial \dot{\tau}}-\Omega=H=ct \ , \tag{47}$$

A new function, the Whittaker function, L^* is defined as follows (if Eq.(44) is also taken into account):

$$L^*=\frac{\partial \Omega}{\partial \dot{\delta}_1}=\frac{\partial \Omega}{\partial \dot{\tau}}=\frac{\partial L}{\partial \dot{\delta}_1}+\sum_{j=2}^{p}\frac{\partial L}{\partial \dot{q}_j}\frac{1}{\dot{\tau}}\dot{q}_j \ . \tag{48}$$

The energy integral written in the form Eq.(47) is derived with respect to δ_i and $\dot{\delta}_i$, taking into account that the function depends on the following variables: $\tau, q_2, q_3, \ldots \ \ldots q_p, \ \ \dot{\tau}, q'_2, q'_3, \ldots \ \ldots q'_p$ as well as the fact that from the above derived relation, it can be expressed $\dot{\tau}$ in terms of the other variables.

The expressions for the partial derivatives of the energy integrals are as follows:

$$\left(\frac{\partial}{\partial \delta_i}\right) \qquad \dot{\tau}\left(\frac{\partial^2 \Omega}{\partial \dot{\tau}\partial \delta_i}+\frac{\partial^2 \Omega}{\partial \dot{\tau}^2}\frac{\partial \dot{\tau}}{\partial \delta_i}\right)-\frac{\partial \Omega}{\partial \delta_i}=0 \tag{49}$$

$$\left(\frac{\partial}{\partial \delta'_i}\right) \qquad \dot{\tau}\left(\frac{\partial^2 \Omega}{\partial \dot{\tau}\partial \delta'_i}+\frac{\partial^2 \Omega}{\partial \dot{\tau}^2}\frac{\partial \dot{\tau}}{\partial \delta'_i}\right)-\frac{\partial \Omega}{\partial \delta'_i}=0 \tag{50}$$

The partial derivatives with respect to and of the function L*, defined according to the relation Eq.(48), are calculated, resulting in:

$$\frac{\partial L^*}{\partial \delta_i}=\frac{\partial^2 \Omega}{\partial \dot{\tau}\partial \delta_i}+\frac{\partial^2 \Omega}{\partial \dot{\tau}^2}\frac{\partial \dot{\tau}}{\partial \delta_i} \tag{51}$$

$$\frac{\partial L^*}{\partial \delta'_i}=\frac{\partial^2 \Omega}{\partial \dot{\tau}\,\partial \delta'_i}+\frac{\partial^2 \Omega}{\partial \dot{\tau}^2}\frac{\partial \dot{\tau}}{\partial \delta'_i} \tag{52}$$

Comparing Eq.(49) with Eq.(51) and Eq.(50) with Eq.(52) we get:

$$\dot{\tau}\frac{\partial L^*}{\partial \delta_i}=\frac{\partial \Omega}{\partial \delta_i} \quad ; \quad \dot{\tau}\frac{\partial L^*}{\partial \delta'_i}=\frac{\partial \Omega}{\partial \delta'_i} \quad . \tag{53}$$

According to relations (39) and (43) it was shown that the following relations are valid:

$$\frac{\partial L}{\partial \delta_i}=\frac{\partial \Omega}{\partial \delta_i} \quad , \quad i=\overline{1,n} \quad ; \tag{54}$$

$$\frac{\partial L}{\partial \dot{\delta}_i}=\frac{1}{\dot{\tau}}\frac{\partial \Omega}{\partial \delta'_i} \quad , \quad i=\overline{2,n} \quad . \tag{55}$$

The system of Lagrange differential equations is considered except for the first equation (this has no significance in this case when it was considered that the first coordinate q_1 is the independent variable τ):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}_i}\right)-\frac{\partial L}{\partial \delta_i}=0 \quad ; \quad i=\overline{2,p} \quad . \tag{56}$$

Considering (55) and (54), equation (56) becomes:

$$\frac{d}{dt}\left(\frac{1}{\dot{\tau}}\frac{\partial \Omega}{\partial \delta'_i}\right)-\frac{\partial \Omega}{\partial \delta_i}=0 \quad ; \quad i=\overline{2,p} \ . \tag{57}$$

Using the relations (53) after substituting them in the previous relation, we obtain:

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \delta'_i}\right)-\dot{\tau}\frac{\partial L^*}{\partial \delta_i}=0 \quad ; \quad i=\overline{2,p} \quad . \tag{58}$$

By writing:

$$\frac{dq_1}{dt}=\frac{\partial \tau}{\partial t}=\dot{\tau} \quad , \tag{59}$$

the equations (58) will have the final form:

$$\frac{d}{d\tau}\left(\frac{\partial L^*}{\partial \delta'_i}\right)-\frac{\partial L^*}{\partial \delta_i}=0 \quad ; \quad i=\overline{2,p} \tag{60}$$

The equations obtained (60) are called Whittaker's equations.

5. CONCLUSIONS

The use of FEM in the dynamic study of elastic MBS is a method that is becoming increasingly common in the analysis of extremely complex industrial applications. Deriving the evolution equations is difficult because of the system's complexity and special features. These can be obtained using a number of comparable classical formalisms. There is a strong need to learn more about these mechanical systems because it is common practice in industrial applications to analyze MBS with planar motion. This study examines the modeling and description of these kinds of systems using various analytical mechanics methodologies. Discussions have been held regarding the Lagrange, GA, Hamilton, Kane, and Maggi formalisms. The methods discussed are obviously interchangeable, and it is up to the researchers to determine which one, given their training and expertise, would be most appropriate for a given application.

In addition to these methods, which bring considerable advantages in certain situations encountered in practice, Whittaker's method can also be considered and used. The need to cope with complex mechanical systems that operate at high speeds and in difficult environments explains the interest in alternate approaches to characterizing their behavior, such as Whittaker's method.

Therefore, it may be said that for the modeling and analysis of complex systems, alternative analytical mechanics techniques, such as the Whittaker equations, should be taken into account. They might be intriguing options to deal with the difficulties brought on by the current technological environment because of the advantages demonstrated in particular applications that call for laborious numerical analysis. However, we note that the application of this method, feasible for FEM applications, does not offer great computational advantages except for simple systems. The number of differential equations is reduced by one, which is not a significant advantage for small mechanical systems encountered in current engineering applications. We estimate some advantages of the method only for applications with a small number of finite elements, for research purposes, and not for engineering applications in the real world.

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