Another Approach for Obtaining the Eigenvalues of a Bernoulli-Euler Beam with a Single In-span Elastic Rod with a Tip Mass

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Abstract: - This paper presents a different approach for obtaining the eigenvalues of a combined system investigated previously. The mechanical vibration system in question consists of a horizontal Bernoulli-Euler beam carrying an axially vibrating rod with a tip mass at an intermediate location. In the context of the application of the Galerkin's procedure and assumed modes approach, the eigenfunctions of the fixed-free rod were previously used as trial functions in a study from the same authors [1]. As an alternative approach, the eigenfunctions of the rod with a tip mass are used as trial functions in this study. Using these eigenfunctions causes very fast convergence with a few numbers of trial functions to determine the eigenvalues. This fact is clearly demonstrated by means of tables and graphs as examples of Bernoulli-Euler beams with different boundary conditions. Also, the numerical results are compared with the values found by the finite element method (FEM), which can be considered as the exact eigenfrequencies of the system. This novel strategy accelerates the convergence rates. The efficacy of the method is demonstrated numerically. In addition, this study successfully demonstrates for the first time that when a continuous system is attached to the horizontal beam, the combined system has new eigenvalues between the original pair of the beam frequencies nearest to the eigenvalues of the attached continuous system, i.e., the axially vibrating rod with a tip mass.

Keywords: - Bernoulli-Euler beams, combined systems, tip mass, eigenfunctions, rapid convergence.

1. INTRODUCTION

The free vibration problem of Bernoulli-Euler and Timoshenko beams that have miscellaneous boundary conditions has been frequently investigated by many researchers. In the existing literature, these beams have attachments like point masses, spring-mass secondary systems, and even rods. One can find a humble reference list on this subject in the work [1]. The more recent studies on the subject will be outlined below briefly.

Jin et al. [2] proposed an analytical method for analyzing the free vibration of multi-span Timoshenko beams under arbitrary boundary conditions. They successfully utilized linear springs and rotational springs to simulate the boundary and support forces of multi-span Timoshenko beams. Song et al. [3] studied the transverse vibration of a Bernoulli-Euler beam with end masses and springs. In their work, the exact frequency equation was derived, and natural frequencies and the corresponding mode shapes were obtained. Sinha [4] developed an algorithm to compute the natural frequencies and mode shapes of a Bernoulli-Euler beam with arbitrary nonuniformities and discontinuities. The author presented numerical results for three nonuniform beams with discontinuities: a single-step circular beam, a multi-step rectangular beam, and a beam that is partly uniform and partly tapered. In the study [5], free vibration and stability analysis of a system consisting of a Bernoulli-Euler beam axially loaded by a tendon was investigated. The tendon was attached to the cantilevered beam at the tip as well as in several spanwise locations using mechanical attachment points. Shemshadi et al. [6] studied a beam-based dynamic vibration absorber (DVA) that consisted of an L-shaped beam with a concentrated mass at its free end to attenuate vibrations in a pipeline. In the paper, a mathematical equation was extracted to design the beam DVA using the dimensional analysis (DA) method. In Ref. [7], the exact solution for the free vibration of Bernoulli–Euler beams with an arbitrary number of translational springs at different locations and under various boundary conditions was obtained using the shape function method. The authors solved the differential equation governing the vibration using the variational iteration method and the Laplace transform. The paper [8] was concerned with the problem of free vibration of a Bernoulli-Euler beam supported with an arbitrary number of translational springs of different constant stiffnesses. The natural frequencies and mode shapes were calculated using the Green's function method. The authors also investigated the effect of seven different boundary conditions at both ends of the beam.

In the authors' previous work [1], a quite different method was used, leading exactly to the same eigenvalue problem obtained in the study [9]. In both studies, the assumed modes method is used to obtain an approximate and systematic formulation for the eigenfrequencies of a beam with different supports and multiple helical spring–mass attachments. Each helical spring is modeled as an axially vibrating elastic rod. Cha et al. [9] used an assumed solution written as a series of space-dependent trial functions with time-dependent generalized coordinates. These expressions were substituted into the kinetic and potential energies, and the equations of motion were obtained using Lagrange's equations. In the study [1], the differential equations of motion of the vibrating system were written more or less directly by making use of the concept of the "characteristic force" [10]. Discretization of the system was then achieved by using the same series solutions as in the work [9], giving in turn the same eigenvalue problem formulation.

In the present study, the assumed modes method is used to obtain eigencharacteristics of the same vibrational system again. As is known, in the assumed modes method, the space-dependent trial functions must satisfy the geometric boundary conditions of the bare linear structure without any attachments. Therefore, the space-dependent trial functions are generally selected as the eigenfunctions of the bare beam and bare rod. The usage of the eigenfunctions of only bare linear structures causes slow convergence. To improve convergence rates, many approaches have been developed. Cha et al. [9] used a spatially linear-varying static mode and free vibration modes together to achieve fast convergence rates. Furthermore, Cha and Park [11] successfully expressed the longitudinal displacement as a sum of eigenfunctions and additional piecewise linear functions and obtained good convergence rates.

In this paper, the space-dependent trial functions for the axially vibrating elastic rod are chosen as the eigenfunctions of a fixed-free rod with a tip mass to accelerate convergence rates in the series solution.

The numerical investigations reveal the fact that although the proposed method in the study [9] and the present method employ quite different functions in series solutions, the convergence rates of the two methods are very close or even identical to each other. In other words, similar convergence rates were successfully achieved by following a quite different way in this study.

It is well known in the literature that attaching a spring–mass system to a beam introduces an additional natural frequency in the combined system. This frequency does not exactly match the natural frequency of the harmonic oscillator but is close to it [12, 13].

Furthermore, Cha and Pierre [14] showed that this property is also valid when a chain of oscillators is attached to another system. This study successfully extends this property by showing that the same behavior can be observed when a continuous elastic system is attached to a beam.

2. THEORY

The authors study the same mechanical system as in [1]. It is a Bernoulli-Euler beam that vibrates transversely with an in-span helical spring-mass system. The spring-mass system is modeled as an axially vibrating rod. To improve numerical performance compared to the previous work, the assumed modes method uses eigenfunctions of a fixed-free rod with a tip mass to describe the displacement of the rod in this work. The mechanical system studied is shown as a cantilevered beam in Figure 1. However, the formulation in this section is quite general and applies to a beam with any boundary conditions.

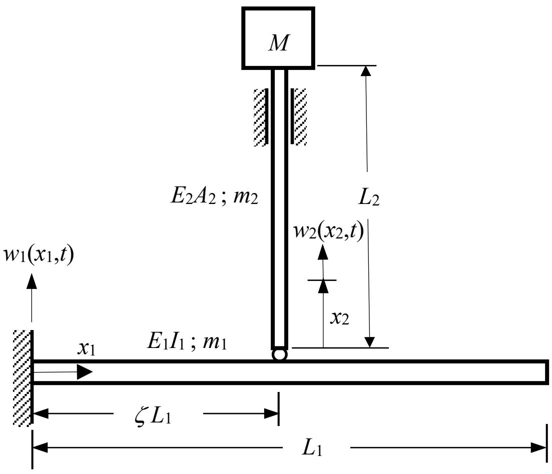

Figure 1. A cantilevered beam carrying an in-span axially vibrating elastic rod with a tip mass

The system includes a cantilevered Bernoulli-Euler beam. An axially vibrating elastic rod is attached to the beam in-span. This rod with a tip mass is a more realistic model of a conventional helical spring-mass system. The equations of motion of the vibrational system and expressions of "characteristic force" are given in expressions (1-3) in the article [1].

As in the assumed modes method, the transverse displacements of the beam can be written as a finite series:

$$w_1(x_1,t)=\sum_{i=1}^{N}\phi_i(x_1)\eta_i(t). \quad (1)$$

Here, N is the number of modes used in the series expansion of the displacement of the beam. The terms $\phi_i(x_1)$ represent the ith eigenfunction of the bare beam. $\eta_i(t)$, are the generalized coordinates to be determined. Similarly, the axial displacements of the rod can be represented as another finite series:

$$w_2(x_2,t)=\sum_{r=1}^{P}\kappa_r(x_2)a_r(t). \quad (2)$$

Here, P is the number of modes used in the series expansion of the axial displacement of the rod. The terms $\kappa_r(x_2)$ are the rth eigenfunction of a fixed-free rod with the tip mass M, and $a_r(t)$, the corresponding generalized coordinates are to be determined. These eigenfunctions are more compatible with the dynamic boundary condition related to the tip mass (See Eq. 17 in [1]). In the assumed modes method, fewer number of selected eigenfunctions gives more accurate results, as will be discussed in the numerical results section.

Considering the bare beam in the mechanical system as a clamped-free beam, the functions $\phi_i(x_1)$ correspond to the mass-normalized eigenfunctions of a clamped-free Bernoulli-Euler beam. These functions are given in Eq. 6 in [1].

Since the vertical rod is attached to the beam, the functions $\kappa_r(x_2)$ are the eigenfunctions of an axially vibrating fixed-free rod with a tip mass. The normalized form of these eigenfunctions is given by:

$$\kappa_r(x_2)=C_r\sin\bar{\beta}_r^*\frac{x_2}{L_2}, \qquad (r=1,2,3,...). \quad (3)$$

The nondimensional eigenfrequency parameter $\bar{\beta}_r^*=\beta_r^*L_2$ for rth mode is obtained by solving the following transcendental equation:

$$\tan\bar{\beta}_r^*=\frac{1}{\bar{\alpha}_M\bar{\beta}_r^*} \quad (4)$$

Details of the derivation are given in Appendix A. The nondimensional mass parameter $\bar{\alpha}_M$ is the ratio between the tip mass and the mass of the vertical rod:

$$\bar{\alpha}_M=\frac{M}{m_2L_2} \quad (5)$$

The coefficients C_r, in the normalized form of the eigenfunctions are found to be (See Appendix A):

$$C_r=\sqrt{\frac{2}{m_2L_2}}\left(1+\bar{\alpha}_M\sin^2\bar{\beta}_r^*\right)^{-1/2} \quad (r=1,2,...) \quad (6)$$

Before proceeding further, it is quite appropriate to give the expressions of the squared eigenfrequencies ω_i^2 of the bare, i.e., clamped-free beam:

$$\omega_i^2=\left(\bar{\beta}_i\right)^4\frac{E_1I_1}{m_1L_1^4}, \qquad (i=1,2,...) \quad (7)$$

The squared eigenfrequencies $\bar{\omega}_i^2$ of the bare, i.e., fixed-free rod with the tip mass are:

$$\bar{\omega}_i^2=\left(\bar{\beta}_i^*\right)^2\frac{E_2A_2}{m_2L_2^2}, \qquad (i=1,2,...) \quad (8)$$

By substituting the series solutions (1) and (2) into the equations of the motion in Eqs. 1-2 in the study [1] and assuming that the functions $\eta_i(t)$ and $a_r(t)$ are exponential in time:

$$\begin{aligned}\eta_i(t)&=\bar{\eta}_i\mathrm{e}^{\lambda t},\\ a_r(t)&=\bar{a}_r\mathrm{e}^{\lambda t},\end{aligned} \quad (9\text{-}10)$$

and then, carrying out the usual Galerkin procedure, we obtain the following equations for the unknowns $\bar{\eta}_i$ and $\bar{a}_r$ [1]:

$$\begin{aligned}&\left(\lambda^2+\omega_i^2\right)\bar{\eta}_i+\left(m_2L_2+M\right)\lambda^2 f_i\left(\sum_{j=1}^{N}f_j\bar{\eta}_j\right)\\ &+\lambda^2 f_i\left(\sum_{r=1}^{P}\left(m_2\overline{\overline{\alpha}}_r+M\kappa_r(L_2)\right)\bar{a}_r\right)=0 \quad (11)\\ &(i=1,2,...,N)\end{aligned}$$

$$\begin{aligned}&\left(\lambda^2+\bar{\omega}_s^2\right)\bar{a}_s+\\ &\lambda^2\left(\sum_{i=1}^{N}\left(m_2\overline{\overline{\alpha}}_s+M\kappa_s(L_2)\right)f_i\bar{\eta}_i\right)=0,\end{aligned} \quad (12)$$

Here, the following abbreviations are used:

$$f_i=\phi_i(\zeta L_1);$$

$$\overline{\overline{\alpha}}_r=\int_0^{L_2}\kappa_r(x_2)dx_2 .$$

where ζL_1 is the attachment location for the rod (See Figure 1).

Eq. (11) was obtained for $\bar{\eta}_i$ is actually the same equation as Eq. (14) in the work [1]. However, the equations for the unknowns $\bar{a}_r$, shown here as Eq. (12) and in the work [1] as Eq. (15), are slightly different. This equation here is simpler. The third term of Eq. (15) in the mentioned work vanishes here, and the equation gets formed as in (12). This simplification arises from using the eigenfunctions of the rod with tip mass in the series expansion of axial displacement of the rod (2). In the assumed modes method, using the eigenfunctions of the rod with a tip mass improves the convergence rate, as shown in the numerical results section. It also simplifies Eq. (12) after Galerkin's procedure.

As mentioned before, the eigenfunctions $\kappa_r(x_2)$ of the axially vibrating rod with tip mass are more compatible with the following dynamical boundary condition of the system:

$$M[\ddot{w}_1(\zeta L_1,t)+\ddot{w}_2(L_2,t)]+ \\ +E_2A_2w_2'(L_2,t)=0 \qquad (13)$$

This selection leads to a more simplified equation (12) than the corresponding equation obtained in the previous study [1].

The following are the auxiliary definitions used in this study:

$$\begin{aligned}
&\bar{\lambda}=\lambda/\omega_0, \quad \alpha_m=\frac{m_2L_2}{m_1L_1}, \quad \alpha_M=\frac{M}{m_1L_1},\\
&\bar{\alpha}_M=\frac{M}{m_2L_2}=\frac{\alpha_M}{\alpha_m}, \quad \alpha_k=\frac{E_2A_2/L_2}{E_1I_1/L_1^3},\\
&\frac{\bar{\omega}_i^2}{\omega_0^2}=\frac{\alpha_k}{\alpha_m}\bar{\beta}_i^{*2}, \quad t_r=\frac{1-\cos\bar{\beta}_r^*}{\bar{\beta}_r^*\sqrt{1+\bar{\alpha}_M\sin^2\bar{\beta}_r^*}},\\
&q_r=\frac{\sin\bar{\beta}_r^{*2}}{\sqrt{1+\bar{\alpha}_M\sin^2\bar{\beta}_r^*}}, \quad \bar{f}_i=f_i\sqrt{m_1L_1},\\
&\kappa_r(L_2)=\sqrt{\frac{2}{m_2L_2}}q_r, \Omega_r\\
&\quad=\sqrt{2\alpha_m}t_r+\sqrt{\frac{2}{\alpha_m}}\alpha_M q_r
\end{aligned} \qquad (14)$$

Using these definitions, the two sets of equations given in (11) and (12) can be written in the following compact forms:

$$\begin{aligned}
&\sum_{j=1}^{N}\left[\left(\alpha_m+\alpha_M\right)\bar{\lambda}^2\bar{f}_i\bar{f}_j+\left(\bar{\lambda}^2+\bar{\beta}_i^4\right)\delta_{ij}\right]\bar{\eta}_j\\
&\quad+\bar{\lambda}^2\bar{f}_i\left(\sum_{r=1}^{P}\Omega_r\bar{a}_r\right)=0, \qquad (i=1,2,\ldots,N)
\end{aligned} \qquad (15)$$

$$\left(\bar{\lambda}^2+\frac{\alpha_k}{\alpha_m}\bar{\beta}_s^{*2}\right)\bar{a}_s+\bar{\lambda}^2\Omega_s\sum_{i=1}^{N}\left(\bar{f}_i\bar{\eta}_i\right)=0 \\ (s=1,2,...,P). \qquad (16)$$

The (N+P) equations above can be combined into the following matrix equation:

$$\begin{bmatrix}[A] & [B]\\ [C] & [D]\end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}\\ \bar{\mathbf{a}}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}. \qquad (17)$$

Here, the following submatrices and additional abbreviations are introduced:

$$\begin{aligned}
&\bar{\mathbf{f}}_{\mathbf{i}_{Px1}}=\left[\bar{f}_i \quad \bar{f}_i \quad \cdots \quad \bar{f}_i\right]^T;\\
&\bar{\mathbf{F}}_{PxN}=\left[\bar{\mathbf{f}}_1 \quad \bar{\mathbf{f}}_2 \quad \cdots \quad \bar{\mathbf{f}}_{\mathbf{N}}\right];\\
&\bar{\mathbf{f}}=\left[\bar{f}_1 \quad \bar{f}_2 \quad \cdots \quad \bar{f}_N\right]^T;\\
&\boldsymbol{\Omega}_{\mathbf{r}_{Nx1}}=\left[\Omega_r \quad \Omega_r \quad \cdots \quad \Omega_r\right]^T;\\
&\boldsymbol{\Omega}_{NxP}=\left[\boldsymbol{\Omega}_\mathbf{1} \quad \boldsymbol{\Omega}_\mathbf{2} \quad \cdots \quad \boldsymbol{\Omega}_\mathbf{P}\right];\\
&\boldsymbol{\eta}=\left[\bar{\eta}_1 \quad \bar{\eta}_2 \quad \cdots \quad \bar{\eta}_N\right]^T;\\
&\bar{\mathbf{a}}=\left[\bar{a}_1 \quad \bar{a}_2 \quad \cdots \quad \bar{a}_P\right],
\end{aligned}$$

$$\begin{aligned}
&[A]=\left(\alpha_m+\alpha_M\right)\bar{\lambda}^2\bar{\mathbf{f}}\bar{\mathbf{f}}^T+\mathbf{diag}\left(\bar{\lambda}^2+\bar{\beta}_i^4\right),\\
&[B]=\bar{\lambda}^2\mathbf{diag}\left(\bar{f}_i\right)\boldsymbol{\Omega},\\
&[C]=\bar{\lambda}^2\mathbf{diag}\left(\Omega_i\right)\bar{\mathbf{F}},\\
&[D]=\mathbf{diag}\left(\bar{\lambda}^2+\frac{\alpha_k}{\alpha_m}\bar{\beta}_i^{*2}\right)
\end{aligned} \qquad (18)$$

The linear, homogeneous matrix equation (17) can now be written as the following generalized eigenvalue problem:

$$\begin{bmatrix}[A''] & [0]\\ [0] & [D'']\end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}\\ \bar{\mathbf{a}}\end{bmatrix}=\lambda^*\begin{bmatrix}[A'] & [B']\\ [C'] & [D']\end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}\\ \bar{\mathbf{a}}\end{bmatrix} \qquad (19)$$

with

$$\begin{aligned}
&\lambda^*=-\bar{\lambda}^2\\
&[A']=\left(\alpha_m+\alpha_M\right)\bar{\mathbf{f}}\bar{\mathbf{f}}^T+[I]_{NxN}\\
&[B']=\mathbf{diag}\left(\bar{f}_i\right)\boldsymbol{\Omega}\\
&[C']=\mathbf{diag}\left(\Omega_i\right)\bar{\mathbf{F}}\\
&[D']=[I]_{PxP}\\
&[A'']=\mathbf{diag}\left(\bar{\beta}_i^4\right)\\
&[D'']=\mathbf{diag}\left(\frac{\alpha_k}{\alpha_m}\bar{\beta}_i^{*2}\right)
\end{aligned} \qquad (20)$$

where $[B']=[C']^T$.

Here, $\left[I\right]_{NxN}$ denotes the N x N identity matrix, and $\left[0\right]$ N x P, or P x N zero matrix.

Solutions of the eigenvalue problem in (19) are $\bar{\lambda}$ values that are purely imaginary numbers in nature, giving the nondimensionalized eigenfrequencies of the combined system in Figure 1. In the next section, it is proved that the same convergence rate of the proposed scheme in the study [9] is obtained here, tracing a quite different way. Numerical results for the various boundary conditions are given in the next section.

One can consult the work [1] for the corresponding expression of the mass-normalized eigenfunctions of Bernoulli-Euler beams $\phi_i\left(x_1\right)$, their nondimensional frequency parameters $\bar{\beta}_i$, and the auxiliary definitions f_i in the cases of a simply-supported beam, a clamped-simply supported beam, and a clamped-clamped supported beam, respectively.

3. NUMERICAL RESULTS

This section represents some numerical results obtained by using the present approach introduced in the previous section. Firstly, the clamped-clamped Bernoulli-Euler beam's nondimensional eigenfrequencies are given in Table 1. It has an in-span connected, axially vibrating rod with a mass positioned on its tip. The values except the last column are taken from the authors' previous work [1]. The values in the last column are calculated by using the present approach derived in this study. All numerical calculations are carried out by using MATLAB. A pseudocode outlining MATLAB implementation is given in Appendix B. Nondimensional physical parameters of the system are chosen as: $\alpha_m = 0.1$, $\alpha_M = 2$, $\alpha_k = 48$, $\zeta = 0.37$. For this numerical case the exact results are not given in the literature, the FEM results are treated as exact results. A mesh convergence study is carried out to check the accuracy of the FEM model in Table 2. The number of elements used in the beam and rod are gradually increased, and the natural frequencies and relative erros given in parathesis are monitored. It is observed that the relative changes in the frequencies become very small as 10^{-3} after 100 elements for the beam and the rod.

The percentage relative errors to exact results are given in parentheses in each cell in Table 1. The P

Table 1. The nondimensional eigenfrequencies $\sqrt{\lambda_i^*}$ of the clamped-clamped Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass. Nondimensional physical parameters of the combined system are taken as: $\alpha_m = 0.1$, $\alpha_M = 2$, $\alpha_k = 48$, $\zeta = 0.37$, and N=10

i	FEM	P=10	P=100	P=200	P=10 (Present Method)
1	4.409433	4.445268 (8.13×10^{-1}%)	4.413158 (8.45×10^{-2}%)	4.411393 (4.44×10^{-2}%)	4.409629 (4.45×10^{-3}%)
2	23.611707	23.666629 (2.33×10^{-1}%)	23.617223 (2.33×10^{-2}%)	23.614499 (1.18×10^{-2}%)	23.611797 (3.81×10^{-4}%)
3	58.809347	59.051877 (4.12×10^{-1}%)	58.835293 (4.41×10^{-2}%)	58.822663 (2.26×10^{-2}%)	58.810122 (1.32×10^{-3}%)
4	73.597601	74.803537 (1.64×10^{0}%)	73.717210 (1.63×10^{-1}%)	73.658964 (8.34×10^{-2}%)	73.601126 (4.79×10^{-3}%)
5	120.868476	120.875343 (5.68×10^{-3}%)	120.869168 (5.72×10^{-4}%)	120.868803 (2.70×10^{-4}%)	120.868477 (8.27×10^{-7}%)
6	138.480971	141.269892 (2.01×10^{0}%)	138.746037 (1.91×10^{-1}%)	138.609411 (9.27×10^{-2}%)	138.473094 (5.69×10^{-3}%)
7	196.192216	197.131589 (4.79×10^{-1}%)	196.286721 (4.82×10^{-2}%)	196.232681 (2.06×10^{-2}%)	196.185484 (3.43×10^{-3}%)
8	211.140764	214.615049 (1.65×10^{0}%)	211.418246 (1.31×10^{-1}%)	211.257168 (5.51×10^{-2}%)	211.102577 (1.81×10^{-2}%)
9	275.843529	281.264754 (1.97×10^{0}%)	276.232554 (1.41×10^{-1}%)	275.965409 (4.42×10^{-2}%)	275.698738 (5.25×10^{-2}%)
10	299.034338	299.403864 (1.24×10^{-1}%)	299.053940 (6.56×10^{-3}%)	299.038503 (1.39×10^{-3}%)	299.042921 (2.87×10^{-3}%)

values represent the number of eigenfunctions of a fixed-free rod carrying the tip mass used in the series expansion (2) for the axial vibrations of the rod. Similarly, N is the number of eigenfunctions used in the series expansion (1) of the bending vibrations of the beam. The investigation of Table 1 clearly reveals the fact that the present method converges to exact results very rapidly. Using only 10 terms in the series expansion for axial vibrations of the rod, a very good numerical approximation for the nondimensional eigenfrequencies $\sqrt{\lambda_i^*}$ of the system is obtained. The present approach, i.e., using eigenfunctions of a fixed-free rod with a tip mass in a series expansion in axial vibrations of the rod, leads to very rapid convergence.

Table 2. Convergence of the first five nondimensional eigenfrequencies $\sqrt{\lambda_i^*}$ of the simply supported Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass with respect to FEM mesh refinement. ($\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$)

Exact [15]	10	100	200	400
5.16587	5.165888 ($3.48\text{x}10^{-4}$)	5.165867 ($5.81\text{x}10^{-5}$)	5.165867 ($5.81\text{x}10^{-5}$)	5.165873 ($5.81\text{x}10^{-5}$)
12.35000	12.350228 ($1.85\text{x}10^{-3}$)	12.350003 ($2.43\text{x}10^{-5}$)	12.350003 ($2.43\text{x}10^{-5}$)	12.350002 ($1.62\text{x}10^{-5}$)
38.90316	38.914505 ($2.92\text{x}10^{-2}$)	38.903239 ($2.03\text{x}10^{-4}$)	38.903181 ($5.40\text{x}10^{-5}$)	38.903168 ($2.06\text{x}10^{-5}$)
51.00988	51.209227 ($3.91\text{x}10^{-1}$)	51.011861 ($3.88\text{x}10^{-3}$)	51.010373 ($9.66\text{x}10^{-4}$)	51.010008 ($2.51\text{x}10^{-4}$)
88.12360	88.262552 ($1.58\text{x}10^{-1}$)	88.124674 ($1.22\text{x}10^{-3}$)	88.123869 ($3.05\text{x}10^{-5}$)	88.123670 ($7.94\text{x}10^{-5}$)

As a second numerical example, Figure 2 shows the effect of the parameter P on the series solution for the first four nondimensional eigenfrequencies of a second system, demonstrating the rapid convergence of the method. The system consists of a simply supported Bernoulli-Euler beam and an in-span attached axially vibrating rod with a tip mass. In Figure 2, the results of the present method are given by the solid lines, and the results of the method introduced in the previous work [1] are given by the dashed lines. The following physical parameters are used: $\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$. For this calculation, the N-value is taken as 10. The value P is more effective than the value N on the convergence of the numerical solutions [1, 9].

The figure reveals the fact that the convergence of the present approach is much better than the previous method. For small values of P, i.e., using a few terms in series solutions (2) with the eigenfunctions of the rod with a tip mass, leads to a very good numerical approximation for the nondimensional eigenfrequencies.

Numerical calculations carried out hereafter are intended to show that the method presented here is equally matched for the work [9] in the context of the convergence rate of the series solutions. As mentioned before, Cha *et al.* [9] proposed to add a static mode to the normal modes in the assumed modes expansion. Tables 3-5 represent the fact that the present method is nearly "identical" to the proposed method in the study [9]. Table 3 shows the nondimensional eigenfunctions of a simply supported beam carrying an axially vibrating rod with a tip mass, where $\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$, and N=10.

Table 3. The nondimensional eigenfrequencies $\sqrt{\lambda_i^*}$ of the simply supported Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass. Nondimensional physical parameters of the combined system are taken as: $\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$ and N=10.

i	Exact [15]	P=10 [1]	N_n=10 (The Proposed Method in [9])	P=10 (The Present Method)
1	5.16587	5.18891 ($4.46\text{x}10^{-1}$%)	5.16598 ($2.13\text{x}10^{-3}$%)	5.16598 ($2.13\text{x}10^{-3}$%)
2	12.35000	12.41126 ($4.96\text{x}10^{-1}$%)	12.35019 ($1.54\text{x}10^{-3}$%)	12.35019 ($1.54\text{x}10^{-3}$%)
3	38.90316	38.95147 ($1.24\text{x}10^{-1}$%)	38.90332 ($4.11\text{x}10^{-4}$%)	38.90333 ($4.37\text{x}10^{-4}$%)
4	51.00988	51.96273 ($1.87\text{x}10^{0}$%)	51.01359 ($7.27\text{x}10^{-3}$%)	51.01358 ($7.25\text{x}10^{-3}$%)
5	88.12360	88.25912 ($1.54\text{x}10^{-1}$%)	88.12461 ($1.15\text{x}10^{-3}$%)	88.12466 ($1.20\text{x}10^{-3}$%)

Furthermore, Figure 3 is provided for better comprehension of the results presented in Table 3. The exact values in the first column are obtained from the work [15]. The values in parentheses are the percentage relative errors. The numerical values in the second, third, and fourth columns represent the results from the previous work [1], the proposed scheme [9], and Eq. (19) of the present study, respectively. Comparing the second and fourth columns clearly indicates that the convergence rates are improved drastically in the present method. Using only 10 terms in the series solution (2), one can obtain a relative percentage error in the order of 10^{-3}-10^{-4}. More importantly, it can be seen that the values in the third and fourth columns are almost identical. In addition, Figure 3 clearly shows that the results presented in this study align very well with those reported in [9]. This reveals the fact that the same convergence rates as in the work [9] are obtained here by utilizing quite a different scheme, i.e., using the eigenfunctions of an axially vibrating rod with a tip mass in series solution.

In Tables 4 and 5, the nondimensional eigenfrequencies of a clamped-free beam and a clamped-simply supported beam are given, respectively. The physical parameters are chosen as $\alpha_m = 0.1$, $\alpha_M = 2$, $\alpha_k = 48$, $\zeta = 0.37$, and N=10. Similarly, Figures 4 and 5 are included to help in the comprehension of the results in Tables 4 and 5, respectively. In the tables, FEM results are treated as exact values. The same trend is also seen in these tables. For only P = 10, the present method gives a very good convergence rate.

Moreover, the authors can obtain the same convergence rates as the work [9] successfully in quite different ways.

As a final numerical result, Tables 6 and 7 demonstrate that the combined system has new eigenfrequencies between the original pair of eigenfrequencies of the horizontal beam closest to the eigenfrequencies of the axially vibrating fixed rod with a tip mass in accordance with the supposition of Cha [16]. In these tables, $\bar{\beta}_i^2$ in the second column represents the appropriate form of the nondimensional frequency parameter for the horizontal bare beam (See Eq. 7). In the third column, $\bar{\lambda}_i^*$ represents the non-dimensional eigenfrequency of the axially vibrating fixed-free rod with a tip mass.

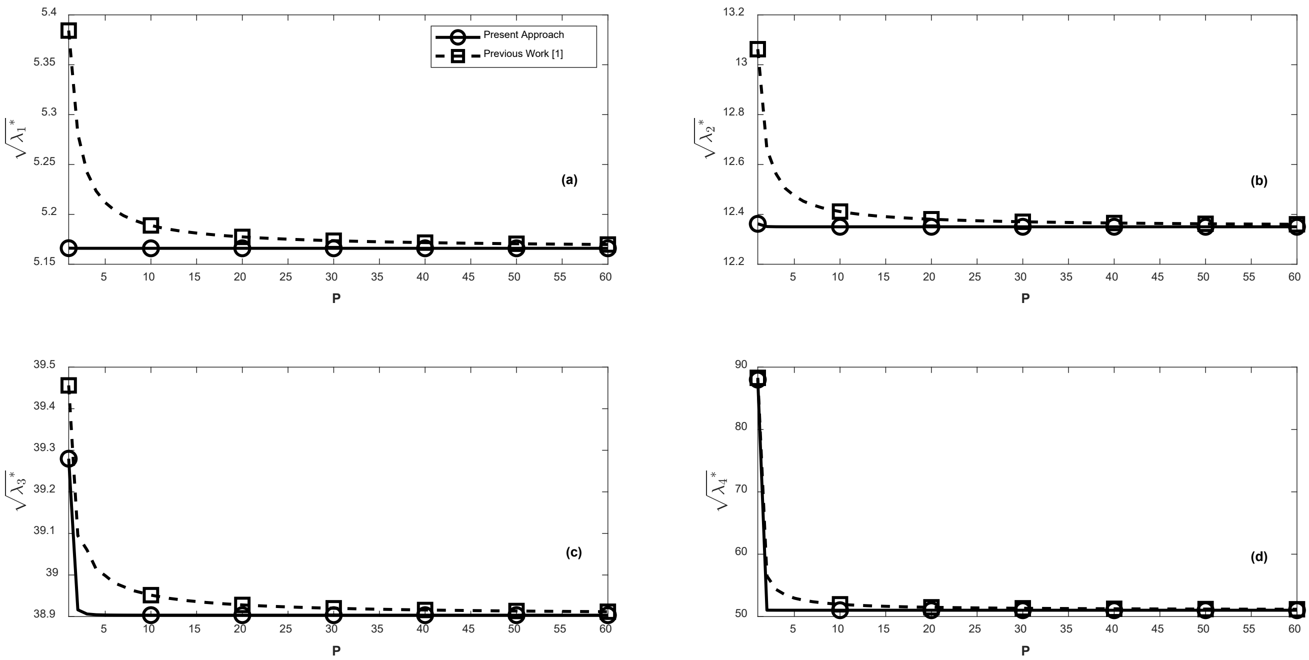

Figure 2. Convergence graphics of the first four nondimensional eigenfrequencies of a simply supported Bernoulli-Euler beam and an in-span attached axially vibrating rod with a tip mass. ($\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$)

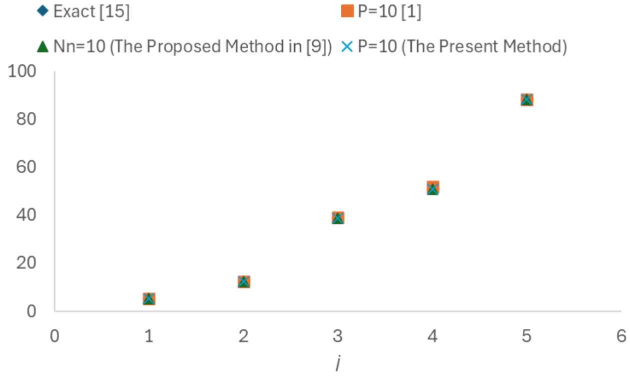

Figure 3. Comparison of the methods given in Table 3.

It can be defined as

$$\bar{\lambda}_i^* = \sqrt{\frac{\alpha_k}{\alpha_m}}\bar{\beta}_i^* \tag{17}$$

with the frequency parameter $\bar{\beta}_i^*$. (See Eqs. 8 and 14). And, in the third column $\sqrt{\lambda_i^*}$ is the nondimensional eigenfrequencies of the combined system. In this column, the shaded values are new eigenfrequencies of the combined system. These values are close to the eigenfrequencies of the axially vibrating fixed rod with a tip mass. Tables 6 and 7 give results for the combined system, consisting of a simply supported beam in Table 6 and a fixed-fixed beam in Table 7, respectively.

Table 4. The nondimensional eigenfrequencies $\sqrt{\lambda_i^*}$ of the clamped-free Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass. Nondimensional physical parameters of the combined system are taken as: $\alpha_m = 0.1$, $\alpha_M = 2$, $\alpha_k = 48$, $\zeta = 0.37$ and N=10.

i	**FEM**	**P=10 [1]**	**N_n=10 (The Proposed Method in [9])**	**P=10 (The Present Method)**
1	2.81602	2.82044 (1.57x10^{-1}%)	2.81606 (1.42x10^{-3}%)	2.81606 (1.42x10^{-3}%)
2	5.52093	5.55783 (6.68x10^{-1}%)	5.52125 (5.80x10^{-3}%)	5.52125 (5.80x10^{-3}%)
3	23.20342	23.25421 (2.19x10^{-1}%)	23.20355 (5.60x10^{-4}%)	23.20355 (5.60x10^{-4}%)
4	58.77136	59.01905 (4.21x10^{-1}%)	58.77280 (2.45x10^{-3}%)	58.77283 (2.50x10^{-3}%)
5	73.62860	74.83547 (1.64x10^{0}%)	73.63581 (9.79x10^{-3}%)	73.63581 (9.79x10^{-3}%)

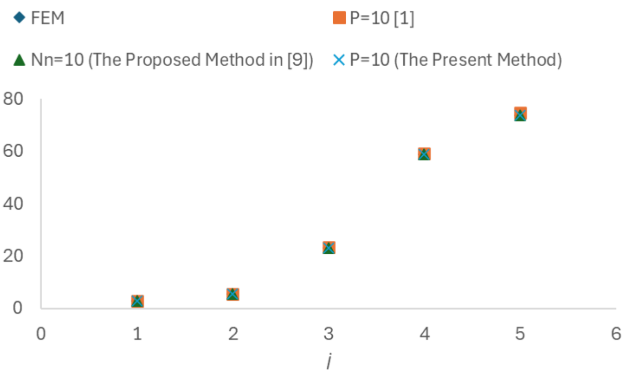

Figure 4. Comparison of the methods given in Table 4.

Table 5. The nondimensional eigenfrequencies $\sqrt{\lambda_i^*}$ of the clamped-simply supported Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass. Nondimensional physical parameters of the combined system are taken as: $\alpha_m = 0.1$, $\alpha_M = 2$, $\alpha_k = 48$, $\zeta = 0.37$ and N=10.

i	**FEM**	**P=10 [1]**	**N_n=10 (The Proposed Method in [9])**	**P=10 (The Present Method)**
1	4.23075	4.26199 (7.38x10^{-1}%)	4.23093 (4.25x10^{-3}%)	4.23093 (4.25x10^{-3}%)
2	16.98748	17.03098 (2.56x10^{-1}%)	16.98762 (8.24x10^{-4}%)	16.98762 (8.24x10^{-4}%)
3	48.43450	48.55227 (2.43x10^{-1}%)	48.43471 (4.34x10^{-4}%)	48.43473 (4.75x10^{-4}%)
4	72.29626	73.63633 (1.85x10^{0}%)	72.30070 (6.14x10^{-3}%)	72.30069 (6.13x10^{-3}%)
5	104.24564	104.25149 (5.61x10^{-3}%)	104.24565 (9.59x10^{-6}%)	104.24565 (9.59x10^{-6}%)

This new approach accelerates the convergence rates. Numerical results reveal the method's effectiveness, showing additionally that it can speed up the convergence rates under different boundary conditions and for various combinations of physical system parameters. For the first time, the present study reveals that the combined system exhibits new eigenfrequencies that lie between the original eigenfrequencies of the horizontal beam, closest to those of the axially vibrating fixed rod with a tip mass.

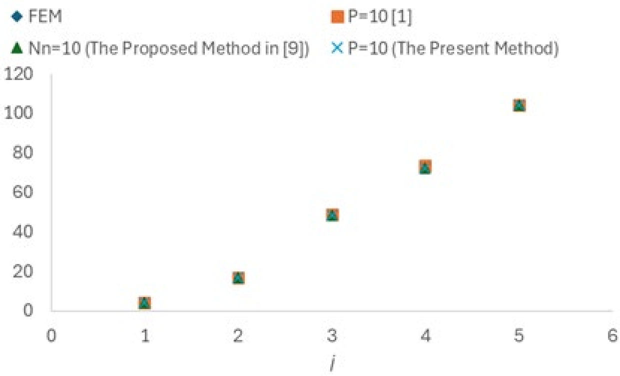

Figure 5. Comparison of the methods given in Table 5.

Moreover, since the presented approach simplifies the structure of the eigenvalue problem, it can be easier to implement it algorithmically on a computer.

The effect of the number of modes N used in the series expansion of the beam displacement in Eq. (1) is shown in Figure 6. The effect of the beam mode number on the first four eigenvalue parameters is found to be minimal, since the selected beam mode shapes exactly satisfy the boundary conditions.

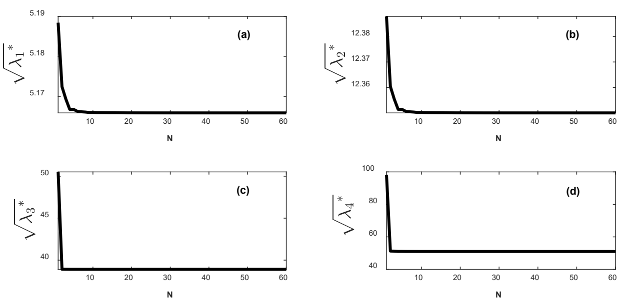

Figure 6. Effect of beam mode number *N* on the first four nondimensional eigenfrequencies of a simply supported Bernoulli-Euler beam and in-span attached axially vibrating rod with a tip mass. $\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$ and P=10.

In Figure 7, the effect of the nondimensional stiffness parameter on the first four nondimensional eigenfrequencies of the system with various boundary conditions is investigated. For all boundary conditions, an increase in the stiffness parameter causes an increase in the eigenfrequencies. This increase is more pronounced for systems incorporating fixed-fixed supported Bernoulli-Euler beams.

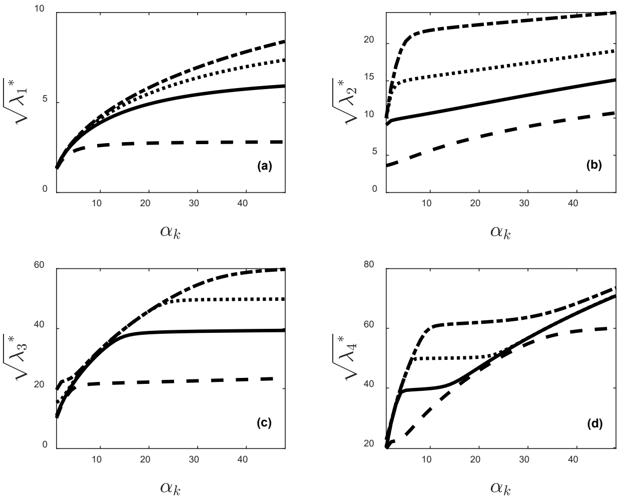

Figure 7. Effect of the nondimensional stiffness parameter on the first four non-dimensional eigenfrequencies of Bernoulli-Euler beams with various boundary conditions and in-span attached axially vibrating rod with a tip mass. (Solid (-): Simply supported; dashed (--): Fixed-free ; dotted (…): Fixed-simply supported ; dash-dot(-.): Fixed-fixed)

Table 6. The first ten nondimensional eigenfrequencies for the bare horizontal simply supported beam, the axially vibrating fixed-free rod with a tip mass, and the combined system. The shaded values are new eigenfrequencies of the combined system. Nondimensional physical parameters are taken as: $\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$ and N=10, P=10

i	$\bar{\beta}_i^2$	$\bar{\lambda}_i^*$	$\sqrt{\lambda_i^*}$
1	9.86960	6.70554	5.16598
2	39.47842	49.63515	12.35019
3	88.82644	97.82918	38.90333
4	157.91367	146.33600	51.01358
5	246.74011	194.92361	88.12466
6	355.30576	243.54380	98.98494
7	483.61062	292.18034	145.22066
8	631.65468	340.82624	159.23632
9	799.43796	389.47799	195.83679
10	986.96044	438.13365	243.96435

Table 7. The first ten nondimensional eigenfrequencies for the bare horizontal fixed-fixed beam, the axially vibrating fixed-free rod with a tip mass, and the combined system. The shaded values are new eigenfrequencies of the combined system. Nondimensional physical parameters are taken as: $\alpha_m = 0.1$, $\alpha_M = 0.5$, $\alpha_k = 24$, $\zeta = 0.6$ and N=10, P=10.

i	$\bar{\beta}_i^2$	$\bar{\lambda}_i^*$	$\sqrt{\lambda_i^*}$
1	22.37329	6.70554	6.31405
2	61.67282	49.63515	22.77036
3	120.90339	97.82918	49.82078
4	199.85945	146.33600	62.45560
5	298.55554	194.92361	98.41960
6	416.99079	243.54380	120.92726
7	555.16525	292.18034	146.67010
8	713.07892	340.82624	192.33064
9	890.73180	389.47799	202.75030
10	1088.12389	438.13365	244.35747

4. CONCLUSIONS

Many studies dealt with the mechanical vibrations of combined systems, which were generally modeled as Bernoulli-Euler beams carrying in-span spring-mass secondary systems. However, in most of these works, the helical springs were assumed to be massless. It is more realistic to model the spring-mass systems as axially vibrating rods with a tip mass. The assumed modes method in connection with the Galerkin’s procedure is commonly used to obtain the eigenvalues of these systems.

To a great extent, the eigenfunctions of only bare beams or rods are used as trial functions in the assumed modes method. While widely used, this approach has a significant drawback. Since these functions do not satisfy the dynamic boundary conditions exactly, satisfactory results can only be obtained using too many modes, leading to a slow convergence rate.

The author’s previous article [1] suffered from the above-mentioned weakness. Recognizing this issue, the authors present a new approach in this study. They use the space-dependent trial functions for the axially vibrating elastic rod in the assumed modes method, the eigenfunctions of a fixed-free rod with a tip mass, through which the satisfaction of the essential dynamic boundary conditions is noticeably better.

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APPENDIX A

In this section, the mass-normalized eigenfunctions of an axially vibrating fixed-free rod with tip mass M are derived. The axial rigidity, length, and mass per unit length of the rod are EA, L, and m, respectively. The equation of motion of the rod is the well-known equation

$$EAu''(x,t) - m\ddot{u}(x,t) = 0, \qquad \text{(A.1)}$$

where $u(x,t)$ represents the axial displacement of the rod at the location x and time t. Primes and dots denote partial derivatives with respect to x and t, as usual. The corresponding boundary conditions are obtained in the following form:

$$u(0,t) = 0,$$
$$EAu'(L,t) + M\ddot{u}(L,t) = 0. \qquad \text{(A.2-3)}$$

Assuming a solution in the harmonic form

$$u(x,t) = U(x)\cos\omega t \qquad \text{(A.4)}$$

where $U(x)$ and ω denote an amplitude function and the corresponding eigenfrequency. When one substitutes the solution (A.4) into the equations (A.1)-(A.3), an ordinary differential equation of order two for $U(x)$ with corresponding boundary conditions are obtained. The general solution of the ordinary differential equation is

$$U(x) = C_1 \sin\beta x + C_2 \cos\beta x \qquad \text{(A.5)}$$

where C_1 and C_2 are integral constants. The corresponding boundary conditions lead to $C_1 = 0$ and the following characteristic equation

$$\tan\bar{\beta} = \frac{1}{\alpha_M \bar{\beta}} \qquad \text{(A.6)}$$

with the following abbreviations

$$\beta^2 = \frac{m\omega^2}{EA}, \quad \bar{\beta} = \beta L, \quad \alpha_M = \frac{M}{mL}. \qquad \text{(A.7)}$$

The r^th eigenfunction of the rod can be written as:

$$U_r = C_r \sin\bar{\beta}_r \frac{x}{L} \qquad \text{(A.8)}$$

where U_r can be normalized by using the following expressions [17]:

$$\int_0^L mU_r^2(x)dx + MU_r^2(L) = 1 \qquad \text{(A.9)}$$

Substituting the eigenfunctions (A.8) into the normalization expression (A.9) and after performing some calculations, one can obtain the following expression for the coefficients C_r

$$C_r = \sqrt{\frac{2}{mL}}\left(1 + \alpha_M \sin^2\bar{\beta}_r\right)^{-1/2} \qquad \text{(A.10)}$$

Now, with these coefficients C_r, the eigenfunctions U_r satisfy the normalization expression given in (A.9).

APPENDIX B

The following pseudocode represents the MATLAB implementation of the matrix eigenvalue problem used in the numerical calculations:

1. Start

2. Load $\bar{\beta}_r^* = \beta_r^* L_2$ (the nondimensional eigenfrequency parameters for the rod with tip mass in (4))

3. Set parameters:
 - N: Number of beam modes in (1)
 - P: Number of rod modes in (2)
 - α_m : Nondimensional mass parameter in (14)
 - α_M : Nondimensional mass parameter in (14)
 - α_k : Nondimensional stiffness parameter in (14)
 - ζ : Attachment point of the rod on the beam
 - $\bar{\alpha}_M = \alpha_M / \alpha_m$

4. Initialize zero matrices:
 - $\bar{\mathbf{F}}_{PxN}$
 - $\mathbf{\Omega}_{NxP}$

5. Depending on the beam boundary condition:
 -Load corresponding nondimensional eigenparameters of the beam $\bar{\beta}_i$
 - For each beam mode k = 1 to N:
 - Compute eigenfunction $\phi_k(x_1)$ value at ζ : $\bar{f}_k$
 - Assign $\bar{\mathbf{F}}(:,k) = \bar{f}_k = \phi_k(\zeta)$

6. For each rod mode k = 1 to P:
 - Compute $\mathbf{\Omega_r}$ using t_k and q_k in (14) based on $\bar{\beta}_k^*$ and $\bar{\alpha}_M$
 - Assign $\mathbf{\Omega}(:,k) = \mathbf{\Omega}_k$ in (18)

7. Construct system matrices in (20):
 - $[A']$
 - $[A'']$
 -$[B']$
 - $[C']$
 - $[D']$
 - $[D'']$

8. Form generalized eigenvalue problem in (19):
 - A = [$[A']$, 0 ; 0 , $[D'']$]
 - B = [$[A']$, $[B']$; $[C']$, $[D']$]

9. Solve (A - λB)v = 0
 - Use eig(A,B)
 - Sort eigenvalues
 - Compute λ = sqrt(-eigenvalue)
 - Keep imaginary part as nondimensional eigenfrequencies: lambdac = imag(λ)

10. Display nondimensional eigenfrequencies.

11. End