# Dynamic Stability of a Conical Pipe, Conveying Fluid and Resting on a Winkler Elastic Foundation

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*Abstract:* - This paper presents an application of the Generalized Differential Quadrature Method (GDQM) to analyze the stability problem of a fluid-conveying conical pipe, clamped at both ends and resting on a Winkler elastic foundation. The main focus is on quantifying the impact of the rigidity of the elastic foundation on the critical fluid velocity, which is the velocity at which the pipe loses stability. A numerical solution was performed for a straight pipe conveying fluid with specified geometric and physical characteristics, where the rigidity of the elastic foundation and the density of the conveyed fluid were considered as parameters. The obtained results, visualized through graphical relationships between the critical velocity and the foundation's rigidity for various densities of the fluid, demonstrate the significant effect of elastic supports on the vibrational behavior and stability of the pipe. The validity of the presented GDQM approach is confirmed by comparison with results obtained using the Transfer Matrix Method (TMM), showing good agreement.
The contribution of this paper lies in presenting a methodology for solving the differential equation describing the lateral displacements of conical piping systems, as well as in a critical assessment of the advantages and disadvantages of GDQM compared to established methods for the dynamic analysis of fluid-conveying pipes.

*Keywords:* - conical pipe, fluid, circular frequency, dynamic stability, flow velocity, GDQM

## 1. INTRODUCTION

Pipes conveying fluid are used in numerous industrial fields such as chemical, petroleum, aviation, and nuclear. The dynamic stability of fluid-conveying pipes has been a research focus for many scientists. Paidoussis was among the first to address this issue [1], [2].

Mohamed Gaith [3] examined a cantilevered, tapered, slender pipe carrying an inviscid, incompressible fluid. The study utilized Euler-Bernoulli and Hamilton's theories to analyse the pipe's vibrations. The Galerkin method was used to solve the governing differential equation. The research determined the complex natural frequencies of the system for various fluid velocities.

In their works, R. Gregory and M. Paidoussis [4], [5], presented numerical studies and experimental results on the dynamic stability of cantilevered pipes with flowing fluid.

I. Elishakoff [6] explored the dynamics of a pipe conveying fluid and resting on Winkler and rotary elastic foundations. The study examined the impact of the parameters of both foundations on the critical fluid velocity.

Sv. Lilkova-Markova [7] explored the dynamic stability of fluid-conveying pipelines with varying boundary conditions: one end was free, and the other was either clamped, simply supported, or free. A segment of the pipeline was supported by a Winkler elastic foundation, and in each configuration, the critical velocity of the fluid was evaluated.

In a more recent study [8], the stability of fluid-conveying pipes resting on a two-parameter elastic foundation was investigated. The authors employed the Euler–Bernoulli beam theory under multiple boundary conditions and applied the Differential Transform Method. Their results demonstrated that the elastic foundation enhances system stability, with increased foundation parameters leading to higher critical flow velocities.

Another contemporary analysis [9] focused on the dynamic stability of a cantilevered pipe subjected to a laterally distributed load, utilizing the Differential Quadrature Method for the assessment.

Ding and Ji [10] presented a comprehensive review of recent developments in vibration control strategies for fluid-conveying pipelines.

The widespread use of nanoscale tubes in scientific and industrial applications has spurred extensive research activity, as highlighted in [11] and [12]. Due to the complexity and cost of nanoscale experimentation, fluid–structure interactions in carbon nanotubes are frequently modeled using continuum mechanics approaches, such as Euler and Timoshenko beam theories.

This study focuses on a conical tube conducting fluid and supported by a Winkler elastic foundation. The Generalized Differential Quadrature Method is utilized. The findings illustrate how the critical fluid velocity depends on the stiffness of the Winkler elastic foundation for fluids with varying densities.

This paper is organized as follows: initially, we introduce the pipe model, including its static scheme and the governing differential equation for its eigen lateral vibrations. Subsequently, we apply the Generalized Differential Quadrature Method to solve the problem, detailing the derivation of the system's frequency equation. The roots of this equation serve as the basis for conclusions regarding the system's stability. Finally, we present and discuss the numerical results, followed by a summary of key conclusions. To validate our findings, we compare them with numerical results obtained using the Transverse Matrix Method for the same system. Furthermore, we provide our perspective on the main advantages and disadvantages of the Generalized Differential Quadrature Method in comparison to other methods for the dynamic analysis of pipe systems.

## 2. PROBLEM FORMULATION

This study examines the dynamic stability of a conical pipe conveying fluid. Figure 1 illustrates the static scheme of the pipe under consideration.

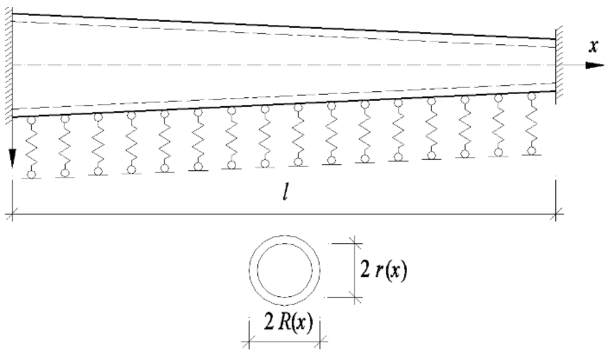


**Figure 1.** Static scheme and cross-section of the investigated pipe, conveying fluid

The transverse vibration of a straight conical pipe, conveying inviscid fluid and resting on a Winkler elastic foundation, is described by the following differential equation, as demonstrated by Yuzhen Z et al. [13].

$$E\frac{\partial^2}{\partial x^2}\left(I(x)\frac{\partial^2 w}{\partial x^2}\right)+\left(m_f+m_p\right)\frac{\partial^2 w}{\partial t^2}+V\frac{d\,m_f}{d\,x}\frac{\partial w}{\partial t}+$$
$$+2V\,m_f\frac{\partial^2 w}{\partial x\partial t}+m_f\frac{\partial w}{\partial t}\frac{dV}{dx}+V^2\frac{d\,m_f}{d\,x}\frac{\partial w}{\partial x}+$$
$$+m_f V^2\frac{\partial^2 w}{\partial x^2}+2\,m_f\,V\,w\frac{dV}{dx}+k_w\,w=0\,. \quad (1)$$

The pipe is made of a linear elastic material characterized by Young's modulus $E$. The cross-sectional parameters include the external radius $R(x)$, internal radius $r(x)$, and the axis moment of inertia $I(x)$. The fluid flowing through the pipe is heavy and incompressible. $w$ is the transverse displacement of the cross-section. The variables $m_f$ and $m_p$ represent the mass of the flowing fluid per unit length of the pipe and the mass of the pipe per unit length, respectively. $V$ denotes the velocity of the flowing fluid, while $k_w$ represents the rigidity of the Winkler elastic foundation.

For the radius of an arbitrary cross-section, the following applies:

$$R(x)=\left(1-\gamma\frac{x}{l}\right)R_0\,;\; r(x)=\left(1-\gamma\frac{x}{l}\right)r_0\,, \quad (2)$$

where $R_0$ and $r_0$ represent the outer and inner radius at the left end of the pipe (inlet cross-section), and $\gamma$ is a reduction coefficient.

The fluid velocity at any cross-section of the pipe is assumed as follows:

$$V(x)=\frac{V_0}{\left(1-\gamma\frac{x}{l}\right)^2}\,, \quad (3)$$

where $V_0=V(0)$ represents the inlet velocity (at the cross-section with an abscissa of $x=0$).

The mass of the pipe $m_p$ at a given abscissa is represented as follows:

$$m_p(x)=\left(1-\gamma\frac{x}{l}\right)^2\rho A(0)\,, \quad (4)$$

where $A(0)$ is the area of the cross-section at the left end of the pipe (the cross-section with an abscissa of $x=0$). $\rho$ is the density of the material of the pipe.

Similarly, the mass of the fluid per unit length of the pipe $m_f$ at a given abscissa is represented as follows:

$$m_f\left(x\right)=\left(1-\gamma\frac{x}{l}\right)^2\rho_f\,A_f\left(0\right), \qquad (5)$$

where $A_f\left(0\right)$ is the cross-section of the fluid at the left end of the pipe (the cross-section with an abscissa of $x=0$). $\rho_f$ is the density of the conveyed fluid.

For the inertial moment $I\left(x\right)$ at any cross-section of the pipe, one has:

$$I\left(x\right)=\left(1-\gamma\frac{x}{l}\right)^4 I\left(0\right), \qquad (6)$$

where $I\left(0\right)$ is the inertial moment of the cross-section at the left end of the pipe.

Equation (1) holds true given the following assumptions [1], [2]:

- The transported fluid is assumed to be heavy, inviscid, and incompressible. This allows for the assumption of uniform fluid velocity across the cross-section;
- Dissipation and damping effects are assumed to be negligible;
- Axial inextensibility of the pipe is assumed;
- Under the premise that the Euler-Bernoulli hypothesis is valid, plane cross-sections, which were originally normal to the pipe's axis, are assumed to remain plane and normal to the deformed axis;
- The pipe material is considered linear elastic, and therefore follows Hooke's law;
- The pipe's length is significantly greater than its characteristic cross-sectional size
- Rotary inertia effects are not considered;

The solution to equation (1) is sought in the form:

$$w\left(x,t\right)=\varphi\left(x\right)e^{\omega t}, \qquad (7)$$

where $\omega$ is the circular frequency.

After transformations, the differential equation (1) assumes the following form:

$$E\frac{d^2}{dx^2}\left(I\left(x\right)\frac{d^2\varphi}{dx^2}\right)+\left(m_f+m_p\right)\omega^2\varphi+V\frac{dm_f}{dx}\omega\varphi+$$
$$+2Vm_f\omega\frac{d\varphi}{dx}+m_f\omega\varphi\frac{dV}{dx}++V^2\frac{dm_f}{dx}\frac{d\varphi}{dx}+$$
$$+m_fV^2\frac{d^2\varphi}{dx^2}+2m_fV\varphi\frac{dV}{dx}+k_w\varphi=0. \qquad (8)$$

The governing equation of motion (8) can be transformed into a system of algebraic equations using the Generalized Differential Quadrature Method (GDQM) (Soltani P. and Saadati M. [14]; Tornabene F. et al. [15]). The core concept of this method involves approximating the derivative of a function at any discrete point within a domain as a weighted linear sum of function values at all discrete points, mathematically expressed as follows:

$$\left.\frac{d^n\varphi\left(x\right)}{dx^n}\right|_{x=x_i}=\sum_{j=1}^{m}\beta_{ij}^{(n)}\varphi\left(x_j\right),\quad i=1,...,m, \qquad (9)$$

where $m$ is the total number of sampling points on the chosen grid along the pipe's axis, and $\beta_{ij}^{(n)}$ is the weighting coefficient corresponding to the $n$-th order derivative at the point $i$.

The present paper adopts the Chebyshev-Gauss-Lobatto point distribution (Tornabene F. et. al. [15]).

$$x_i=\frac{l}{2}\left[1-\cos\left(\frac{i-1}{m-1}\pi\right)\right],\quad i=1,...,m \qquad (10)$$

The weighting coefficients are determined using Lagrange interpolation functions. For the first derivative, the weighting coefficients are computed as per Tornabene F. et al. [15]:

$$\beta_{ij}^{(1)}=\frac{L^{(1)}\left(x_i\right)}{\left(x_i-x_j\right)L^{(1)}\left(x_j\right)},\quad i,j=1,...,m,i\neq j \qquad (11)$$

$$x_i=\frac{l}{2}\left[1-\cos\left(\frac{i-1}{m-1}\pi\right)\right],\quad i=1,...,m, \qquad (12)$$

while for higher-order derivatives, the coefficients are obtained iteratively:

$$\beta_{ij}^{(n)}=n\left(\beta_{ii}^{(n-1)}\beta_{ij}^{(1)}-\frac{\beta_{ij}^{(n-1)}}{x_i-x_j}\right),$$
$$i,j=1,...,m;i\neq j;n=2,..,\left(m-1\right). \qquad (13)$$

$$\beta_{ii}^{(n)}=-\sum_{j=1,j\neq i}^{m}\beta_{ij}^{(n)},$$
$$i,j=1,...,m;\; n=2,..,\left(m-1\right). \qquad (14)$$

The first derivative of Lagrange interpolating polynomials in equation (11) is defined at each point $x_k$ as

$$L^{(1)}\left(x_k\right)=\prod_{l=1,l\neq k}^{m}\left(x_k-x_l\right),\quad k=1,...,m \qquad (15)$$

Using Lagrange interpolating polynomials together with Chebyshev-Gauss-Lobatto sampling points from equation (10) guarantees convergence, thereby reducing error as the number of sampling points increases.

Using the Generalized Differential Quadrature Method (GDQM), the governing equation (8) is transformed into a discrete form at the points $i=3,4,...,(m-2)$.

$$EI(x_i)\sum_{j=1}^{m}\beta_{ij}^{(4)}\varphi(x_j)+2E\frac{dI(x_i)}{dx}\sum_{j=1}^{m}\beta_{ij}^{(3)}\varphi(x_j)+$$

$$+E\frac{d^2I(x_i)}{dx^2}\sum_{j=1}^{m}\beta_{ij}^{(2)}\varphi(x_j)+$$

$$+\left[m_f(x_i)+m_p(x_i)\right]\omega^2\varphi(x_j)+$$

$$+V(x_i)\frac{dm_f(x_j)}{dx}\omega\varphi(x_j)+$$

$$+2V(x_i)m_f(x_i)\omega\sum_{j=1}^{m}\beta_{ij}^{(1)}\varphi(x_j)+$$

$$+m_f(x_i)\omega\frac{dV(x_i)}{dx}\varphi(x_j)+$$

$$+V^2(x_i)\frac{dm_f(x_i)}{dx}\sum_{j=1}^{m}\beta_{ij}^{(1)}\varphi(x_j)+$$

$$+m_f(x_i)V^2(x_i)\sum_{j=1}^{m}\beta_{ij}^{(2)}\varphi(x_j)+$$

$$+2m_f(x_i)V(x_i)\frac{dV(x_i)}{dx}\varphi(x_j)+$$

$$+k_w\varphi(x_j)=0 \qquad (16)$$

The boundary conditions for the pipe depicted in Figure 1 are:

$$\varphi(x_1)=\varphi(x_m)=0 \qquad (17)$$

$$\left.\frac{d\varphi(x)}{dx}\right|_{x=x_1}=\left.\frac{d\varphi(x)}{dx}\right|_{x=x_m}=0 \qquad (18)$$

Equation (16), can be rewritten in the following matrix form:

$$\left[B^{(4)}+B^{(3)}+B^{(2)}+B^{(1)}\right]\delta+I\delta_d=0. \qquad (19)$$

In (19)

$$B_{ij}^{(4)}=EI(x_i)\beta_{ij}^{(4)};\quad B_{ij}^{(3)}=2E\frac{dI(x_i)}{dx}\beta_{ij}^{(3)};$$

$$B_{ij}^{(2)}=\left[E\frac{d^2I(x_i)}{dx^2}+m_f(x_i)V^2(x_i)\right]\beta_{ij}^{(2)};$$

$$B_{ij}^{(1)}=\left[2V(x_i)m_f(x_i)\omega+V^2(x_i)\frac{dm_f(x_i)}{dx}\right]\beta_{ij}^{(1)};$$

$$I_{ij}=\left[m_f(x_i)+m_p(x_i)\right]\omega^2+V(x_i)\frac{dm_f(x_i)}{dx}\omega+$$

$$+m_f(x_i)\omega\frac{dV(x_i)}{dx}+2m_f(x_i)V(x_i)\frac{dV(x_i)}{dx}+k_w$$

$$i=3,4,...,(m-2),\ j=1,...,m. \qquad (20)$$

$$i=3,4,...,(m-2),\ j=1,...,m \qquad (21)$$

$$\delta_d=\left\{\varphi(x_3),...,\varphi(x_{m-2})\right\}^T \qquad (22)$$

The four boundary conditions (17) and (18) are also written in a matrix form.

$$K_b\delta=0, \qquad (23)$$

where

$$K_b=\begin{vmatrix}1 & 0 & 0 & ... & 0\\ \beta_{11}^{1} & \beta_{12}^{(1)} & \beta_{13}^{(1)} & ... & \beta_{1m}^{(1)}\\ 0 & 0 & 0 & ... & 1\\ \beta_{m1}^{(1)} & \beta_{m2}^{(1)} & \beta_{m3}^{(1)} & ... & \beta_{mm}^{(1)}\end{vmatrix}. \qquad (24)$$

The discrete field (19) can be integrated with the boundary conditions (23) to form algebraic equations with $m$ unknown nodal displacements as follows:

$$\begin{vmatrix}K_b\\ B^{(4)}+B^{(3)}+B^{(2)}+B^{(1)}\end{vmatrix}\delta+\begin{vmatrix}0\\ I\delta_d\end{vmatrix}=0. \qquad (25)$$

Equation (25) represents an eigenvalue problem. For different values of the inflow velocity $V_0$ and different values of the rigidity of the Winkler elastic foundation $k_w$ are obtained the natural frequencies $\omega$. If $\mathrm{Re}\,\omega<0$ the system is stable. At $\mathrm{Re}\,\omega=0$ the system is at the edge of loss of stability, and the corresponding fluid velocity is the critical fluid velocity $V_{0,cr}$.

## 3. NUMERICAL RESULTS

Numerical studies have been carried out for fluid flowing through pipes, as shown in Figure 1.

The geometric and the material characteristics of the pipes are: the inner and the outer radii of the inlet cross-section of the pipe $r=0.022\ m$ and $R=0.024\ m$, the inner and the outer radii of the outlet cross-section of the pipe $r=0.012\ m$ and $R=0.014\ m$, Young's modulus $E=210\,GPa$, the density of the material of the pipe, and the length of the pipe is $l = 3m$. In the calculation, the number of sampling points is considered $m = 21$.

For the pipe shown in Figure 1, the critical inlet velocities were obtained for different values of the rigidity of the Winkler foundation, considering three types of conveyed fluid. The results are displayed in Figure 2.

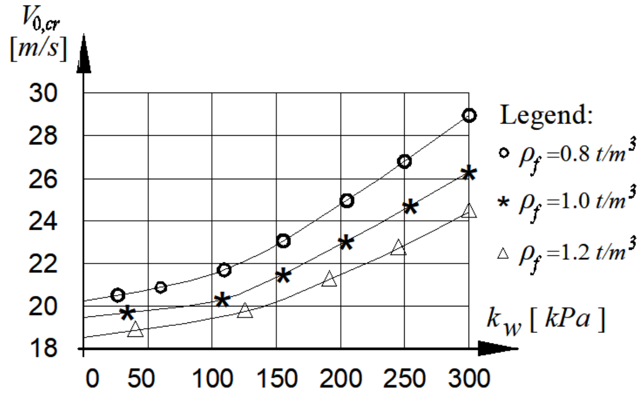


**Figure 2.** Dependence of the critical inlet fluid velocity on the rigidity of the Winkler foundation.

The results obtained via the Generalized Differential Quadrature Method were compared to the Transfer Matrix Method (TMM) solution for the same system. With 10 eigenforms in the TMM, the critical velocities showed an average discrepancy of 5.5%. As anticipated, increasing the TMM eigenforms to 15 lowered the difference between the two methods to 3.1%.

## 4. CONCLUSIONS

The GDQM employed in this paper allows relatively easy determination of the first natural frequencies of the vibrations of conical pipes with flowing fluid. The method could be competitive with other established approaches for investigating the dynamic stability of the pipes conveying fluid, such as the Transfer Matrix Method and the Finite Element Method (FEM).

In the scientific literature, it has been shown that TTM and GDQM have a significant advantage over FEM. This is particularly evident when considering the application of FEM to pipelines with numerous spans, which leads to a substantial increase in computational time due to the expanding order of the system's property matrices. The TMM stands in stark contrast, as its overall transfer matrix retains a fixed order, irrespective of the number of spans. Furthermore, the GDQM offers an efficient approximation of derivatives in the pipe's lateral vibration equation through the utilization of weighted sums of function values at discrete points, resulting in rapid convergence with a sparse grid.

The investigations conducted in the present study not only demonstrate the application of the Generalized Differential Quadrature Method (GDQM) for analyzing the dynamic stability of fluid-conveying conical pipes but also reveal how two key parameters of the pipe-fluid dynamic system - the density of the transported fluid and the stiffness of the Winkler elastic foundation - influence its stability.

Based on the results of the numerical investigations presented in the article (Fig. 2), several major conclusions can be made:

- The results show that the fluid velocity significantly influences the dynamic response of the system, potentially compromising its safety. Since the critical velocity depends on various system parameters - including the density of the transported fluid and the stiffness of the Winkler elastic foundation - pipe operators should ensure that flow velocities remain below the critical threshold to prevent potential damage.
- The stability of the system is inversely related to the density of the fluid - increasing density lowers stability. The results clearly show that, for each of the investigated foundation stiffness values, the critical velocity of the fluid is higher for fluids with lower densities.
- Another key finding is that increasing the stiffness of the Winkler elastic foundation enhances system stability - greater foundation rigidity leads to a higher critical velocity. This implies that the system can withstand higher fluid velocities without experiencing instability.
- When the stiffness of the elastic foundation is below 120 *kPa*, the variation in critical velocity remains gradual. However, beyond 120 *kPa*, the critical velocity increases more significantly. This trend is consistently observed across all three fluids examined.
- As the stiffness of the Winkler foundation increases, the divergence between the critical velocities of the least and most dense fluids becomes more pronounced.

It is worth noting that damping in the Winkler elastic foundation also affects the system's stability; however, this effect is beyond the scope of the present study.

The classical Winkler foundation is widely employed as a modeling approach in geotechnical analyses. In this model, the deflection at any point on the surface of an elastic medium is directly proportional to the load applied at that point and remains unaffected by loads applied elsewhere — a limitation that reduces its realism. To address this shortcoming, the Pasternak foundation model was developed, introducing a two-parameter formulation that better captures the mechanical behavior of the medium. While the present study focuses on the influence of Winkler foundation stiffness and fluid density on the dynamic stability of conical fluid-conveying pipes, the potential impact of Pasternak-type elastic foundations remains an open and promising area for future research.

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