
Some Aspects Concerning the Vibrations of a Nonlinear Mechanical System with a Medical Device

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Abstract: - In this paper, some problems about the parameters of the vibrations (periods, amplitudes, and mean value) for one mechanical system housing a medical device are discussed. The system is acted upon by an earthquake for a certain period, then oscillates freely. The study focuses on both the vibrations during the earthquake and the free oscillations. In addition, the medical device cannot be separated from the last (the third) shell; that is, there exists no motion between this last shell and the medical device. The limitations are determined by the amplitudes, velocities, and accelerations of the medical device's vibrations. The elastic forces in the system are nonlinear. The vibrations are small, so the classical approximation applies. The analytical solution of the equations of motion is also described for a particular case of motion, as well as for the case of an earthquake. The system is considered to have only two degrees of freedom, the vertical displacements of the shells. The analytical solution is obtained using the multiple-scales method. The situations in which the resonance appears are also presented. Some results of the numerical study are also presented. A numerical study shows that rotational displacements have a small influence on the characteristics of the vertical displacement. The influence of different parameters is also highlighted.

Keywords: - analytic solution, numerical approximation, influence of parameters

1. INTRODUCTION

The medical device was presented in a previous paper [1]. The numerical study was performed in a few hypotheses concerning the permanent contact between the medical device and the third shell, the expressions of the elastic forces which are non-linear (the elastic forces are given by polynomial expressions of the deformations, the degrees of the polynomials being odd (for the existence and uniqueness of the equilibrium positions)), and the values of the rotations of the shells which are small ones in order to realize the linearization of the trigonometric functions.

The analysis was performed in the presence of an earthquake [1] and highlights the influence of various parameters and friction. The system under consideration has 6 degrees of freedom, and the results are obtained numerically.

In general, the references highlight two directions of study. The first direction assumes that all

elements are linear and that the general theory of linear systems applies. The second one assumes that at least one element exhibits nonlinear behavior, which significantly complicates the study. In both directions of study, the system may have one or more degrees of freedom (including an infinite number). The nonlinear situation and the response are captured in [2, 3]. In both situations, the systems under consideration have a finite number of degrees of freedom. In [2], the authors use the multiple-scale method for a two-degree-of-freedom system. The excitations are harmonic, and their frequencies vary very slowly over time. The results are validated by numerical analysis. Reference [3] is dedicated to tyre structure. Different modes of vibration are presented. The influences of earthquakes are studied in [4, 5]. The studies are performed using the finite element method [4] or transfer matrices [5]. The earthquake model is based on a real-world event in Taiwan [4]. The building considered in [5] is a 15-story one, while the earthquake is of first

classification, with a duration of 10 seconds. The influence of various parameters, such as beam shape, is presented in [6]. The study is numerical, and the vibrations are free (after the earthquake). The results are validated using the finite element method. The wave propagation is detailed in [7]. Propagation is considered to occur in periodic solid structures and metamaterials. The authors propose a new method to identify the structural vibration mode. Validation is performed using the already published literature. The study of vibration with the finite element method is given in [8]. The authors use the Hamilton principle and the finite element method for every quasi-conforming six-node triangular element. The results are numerically obtained. Vibrations of a tyre are discussed in [9] (see also, reference [3]) using a model of contact (the model is a ring one now, derived from a simply Euler-Bernoulli beam; the eigenmodes are obtained by solving the eigenvalue problem), while vibrations resulting from buckling are described in [10] using vibration correlation technique, while the natural frequency is calculated with the aid of a finite element model.

Other references take into discussion the problems of the minimization of residual vibrations [11] (the structure is a composite one, the method used is the Discrete Material Optimization, while the solution is obtained with the aid of Sequential Linear Programming), weight of mechanical system [12] (the system is a flexible transmission, while the authors use a non-linear mesh force and harmonic balance method; the authors report the detection of bifurcation), suppression of different vibrations [13] with the aid of a flywheel mounted centrifugal pendulum absorber (authors perform a linear analysis), some problems of bifurcations [14] (see also the previous references; the model proposed by the authors [14] is used for the pretwisted composite rotating blades, the results being compared with those given in the references). The vibration frequencies are treated in [15] with the aid of finite elements and a modal stability procedure for a model with twenty nodes, the elements being hexahedral ones, while other authors take into consideration some special properties of the material [16] (the layers are viscoelastic, modeled with the aid of Zener material).

The topology is also used to suppress vibrations [17]. The authors use an eigenvector-guided topology-based optimization method to perform modal control and vibration suppression. The finite element method is used to study the vibrations [18] of beam-like structures. All simple effects are considered (membrane, bending, transverse shear, and torsion). The model size is reduced by eliminating the slave nodes. The particular

vibrations of reactors are described in [19], proving the existence of certain modes.

Some authors describe the steady-state solution [20]. The system is a multi-degree-of-freedom one. Starting from practice, the authors consider that the optimal place to attach the conventional dynamic vibration neutralizer is the last floor of a building. Some aspects concerning an earthquake are also discussed (the transient response). The results are numerically obtained. Other authors study flexible multibody systems [21]. The system under consideration is a flexible multibody system, with vibrations being linear. The authors consider both free and forced vibrations. The elastic elements are bodies or linkages. The method is applied to a machine-tool system. Welded systems are considered in [22]. The authors propose a robust mesh-insensitive method with modal decomposition. The method is compared with existing methods and its advantages highlighted.

The method of study for a system is the application of nonlinear mechanical energy sinks [23] to attenuate passive vibrations of blades. The natural modes are obtained using finite element analysis. The analysis of a prestressed rotating structure using the finite element method is presented in [24]. The authors consider the free-response to compute the vibration modes of the structures and the force response as a sum of eigenvectors. Structures with large displacements are presented in [25]. It is known that the presence of large displacements at specific locations indicates vibration due to energy redistribution. A mistuning ratio is defined, and with the aid of its value, it is predicted how strongly the vibration is confined.

Vibrations are also studied using group theory [26]. The systems are always symmetric. The deformation of such a system is the sum of deformations associated with multiple symmetry species, while the vibration modes are classified into symmetry species. Different sources of vibrations are presented in Reference [27]. The study is performed for a medium-low-speed maglev vehicle. The authors consider active feedback control and investigate stability and bifurcations using dynamical theory and numerical simulations. The same problem is discussed for honeycomb structures [28] from both theoretical and experimental perspectives, considering a new model that accounts for the material nonlinearity of the honeycomb core.

Free vibrations of beams are analyzed in [29, 30]. The authors determine the maximum resonant response [29] and present an analytical solution for the free vibration response. In Reference [30], the free vibrations of 3D Euler-Bernoulli and Timoshenko beams are analyzed.

Some discussions of pipe vibrations are given in [31]. The pipes are affected by base excitation and pulsation. The solution is obtained using a modified transfer-matrix model.

Different methods were proposed for the protection of the distribution of medical gas [32] or for the attenuation of ground vibrations of medical devices, which consist of the creation of some barriers [33] (periodic wave impeding barriers).

Even simple systems [34] may exhibit nonlinear behavior (linearized kinematic relations for an Euler-Bernoulli beam).

For a medical device, some special conditions are imposed: the amplitude of vibrations, the velocities and accelerations of vibrations (they cannot exceed some critical values). Usually, the medical device is linked to a shell. If the medical device is not linked to a shell, then the equations of motion can change their expressions in time (the system may have different degrees of freedom at different intervals of time).

Some authors [35] studied the behavior of a nonlinear dynamic vibration absorber. The study is based on first-order deformation theory, the spectro-geometric method, and the incremental balance method. Other authors [36] present the behavior of conventional tuned liquid dampers with a narrow base under harmonic and seismic excitations. The spring-mass model is considered nonlinear.

Analytical expressions for nonlinear vibrations in certain cases are given in [37].

Because of the very complicated mathematical apparatus, the authors avoid studying such mechanical systems, or, if they do, they consider only a few particular cases.

In general, nonlinear systems are studied using numerical methods or the finite element method. The two approaches validate one another, or they are validated using experimental data (if the systems are simple) or by practical data (obtained from previous earthquakes).

2. MECHANICAL MODEL

The system is captured in Fig. 1. It consists in 3 horizontal shells, the system having 6 degrees of freedom.

It is assumed that one knows the functions $z_1 = z_1(t)$, $\psi_1 = \psi_1(t)$, $\theta_1 = \theta_1(t)$. These functions are given by an earthquake, and they are assumed to be harmonic on a certain interval of time.

The equations of motion of the shells of the system are given by [1]

$$m_2 \ddot{z}_2 = f_{2131}^d - f_{1121}^d + f_{2232}^d - f_{1222}^d + f_{2333}^d - f_{1323}^d + f_{2434}^d - f_{1424}^d - m_2 g, \quad (1)$$

$$J_{x_2}^{(2)} \ddot{\psi}_2 = -[(f_{2131}^d - f_{1121}^d) + (f_{2232}^d - f_{1222}^d) - (f_{2333}^d - f_{1323}^d) - (f_{2343}^d - f_{1424}^d)] l_1, \quad (2)$$

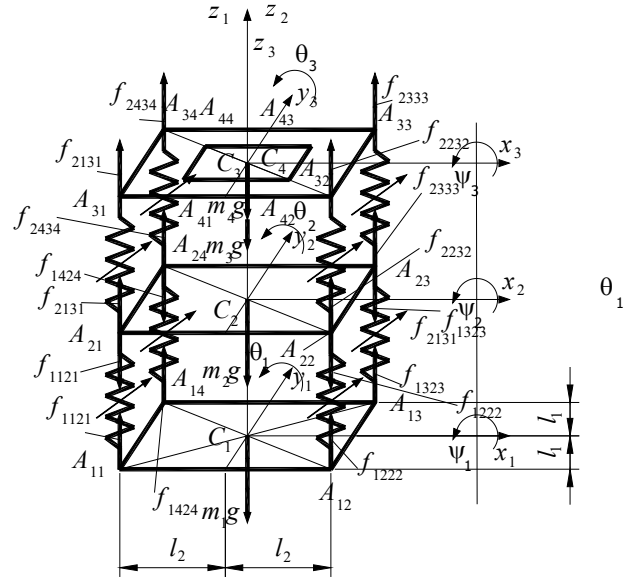


Figure 1. Mechanical system [1]

$$J_{y_2}^{(2)} \ddot{\theta}_2 = -[(f_{2131}^d - f_{1121}^d) + (f_{2232}^d - f_{1222}^d) + (f_{2333}^d - f_{1323}^d) - (f_{2343}^d - f_{1424}^d)] l_2, \quad (3)$$

$$(m_3 + m_4) \ddot{z}_3 = -f_{2131}^d - f_{2232}^d - f_{2333}^d - f_{2434}^d - (m_3 + m_4)g, \quad (4)$$

$$J_{x_3}^{(3)} \ddot{\psi}_3 = (f_{2131}^d + f_{2232}^d - f_{2333}^d - f_{2343}^d) l_1, \quad (5)$$

$$J_{y_3}^{(3)} \ddot{\theta}_3 = (-f_{2131}^d + f_{2232}^d + f_{2333}^d - f_{2434}^d) l_2, \quad (6)$$

where:

$$f_{2131}^d = k_{21} [(z_3 - z_2) - (\psi_3 - \psi_2) l_1 + (\theta_3 - \theta_2) l_2] + k_{21}^* [(z_3 - z_2) - (\psi_3 - \psi_2) l_1 + (\theta_3 - \theta_2) l_2]^3, \quad (7)$$

$$f_{2232}^d = k_{22} [(z_3 - z_2) - (\psi_3 - \psi_2) l_1 - (\theta_3 - \theta_2) l_2] + k_{22}^* [(z_3 - z_2) - (\psi_3 - \psi_2) l_1 - (\theta_3 - \theta_2) l_2]^3, \quad (8)$$

$$f_{2333}^d = k_{23} [(z_3 - z_2) + (\psi_3 - \psi_2) l_1 - (\theta_3 - \theta_2) l_2] + k_{23}^* [(z_3 - z_2) + (\psi_3 - \psi_2) l_1 - (\theta_3 - \theta_2) l_2]^3, \quad (9)$$

$$f_{2434}^d = k_{24}[(z_3 - z_2) + (\psi_3 - \psi_2)l_1 + (\theta_3 - \theta_2)l_2] + k_{24}^*[(z_3 - z_2) + (\psi_3 - \psi_2)l_1 + (\theta_3 - \theta_2)l_2]^3, \quad (10)$$

$$f_{1121}^d = k_{11}[(z_2 - z_1) - (\psi_2 - \psi_1)l_1 + (\theta_2 - \theta_1)l_2] + k_{11}^*[(z_2 - z_1) - (\psi_2 - \psi_1)l_1 + (\theta_2 - \theta_1)l_2]^3, \quad (11)$$

$$f_{1222}^d = k_{12}[(z_2 - z_1) - (\psi_2 - \psi_1)l_1 - (\theta_2 - \theta_1)l_2] + k_{12}^*[(z_2 - z_1) - (\psi_2 - \psi_1)l_1 - (\theta_2 - \theta_1)l_2]^3, \quad (12)$$

$$f_{1323}^d = k_{13}[(z_2 - z_1) + (\psi_2 - \psi_1)l_1 - (\theta_2 - \theta_1)l_2] + k_{13}^*[(z_2 - z_1) + (\psi_2 - \psi_1)l_1 - (\theta_2 - \theta_1)l_2]^3, \quad (13)$$

$$f_{1424}^d = k_{14}[(z_2 - z_1) + (\psi_2 - \psi_1)l_1 + (\theta_2 - \theta_1)l_2] + k_{14}^*[(z_2 - z_1) + (\psi_2 - \psi_1)l_1 + (\theta_2 - \theta_1)l_2]^3. \quad (14)$$

3. ANALYTICAL STUDY

This study is limited to the situation in which the earthquake acts only in the vertical direction, that is, the system has only two degrees of freedom (linear displacement in the vertical direction). The rotations are considered to vanish for all the shells.

In this situation, the system of equations of motion reduces to equations (1) and (4) and

$$f_{2131}^d = k_{21}(z_3 - z_2) + k_{21}^*(z_3 - z_2)^3 \quad (15)$$

$$f_{2232}^d = k_{22}(z_3 - z_2)k_{22}^*(z_3 - z_2)^3 \quad (16)$$

$$f_{2333}^d = k_{23}(z_3 - z_2) + k_{23}^*(z_3 - z_2)^3 \quad (17)$$

$$f_{2434}^d = k_{24}(z_3 - z_2) + k_{24}^*(z_3 - z_2)^3 \quad (18)$$

$$f_{1121}^d = k_{11}(z_2 - z_1) + k_{11}^*(z_2 - z_1)^3 \quad (19)$$

$$f_{1222}^d = k_{12}(z_2 - z_1) + k_{12}^*(z_2 - z_1)^3 \quad (20)$$

$$f_{1323}^d = k_{13}(z_2 - z_1) + k_{13}^*(z_2 - z_1)^3 \quad (21)$$

$$f_{1424}^d = k_{14}(z_2 - z_1) + k_{14}^*(z_2 - z_1)^3 \quad (22)$$

For simplicity, it is assumed that all k_{ij} are equal, $k_{ij} = k_{21}$, $i = \overline{1, 2}$, $j = \overline{1, 4}$, and all k_{ij}^* are also equal $k_{ij}^* = k_{21}^*$, $i = \overline{1, 2}$, $j = \overline{1, 4}$. In this way, one gets

$$f_{2131}^d - f_{1131}^d + f_{2232}^d - f_{1222}^d + f_{2333}^d - f_{1323}^d + f_{2434}^d - f_{1434}^d = 4k_{21}(z_3 - z_2) - 4k_{11}(z_2 - z_1) + 4k_{21}^*(z_3 - z_2)^3 - 4k_{11}^*(z_2 - z_1)^3, \quad (23)$$

$$- f_{2131}^d - f_{2232}^d - f_{2333}^d - f_{2434}^d = -4k_{21}(z_3 - z_2) - 4k_{21}^*(z_3 - z_2)^3. \quad (24)$$

Case 1. Further on, for simplicity, one will take $z_1(t) = 0$ (the earthquake is absent, but this is only for the analytical presentation). This situation is presented after the earthquake.

One is looking for the solution in the form

$$z_2 = \varepsilon z_{21}(T_0, T_2, \dots) + \varepsilon^3 z_{23}(T_0, T_2, \dots) + \dots \quad (25)$$

$$z_3 = \varepsilon z_{31}(T_0, T_2, \dots) + \varepsilon^3 z_{33}(T_0, T_2, \dots) + \dots \quad (26)$$

where $T_n = \varepsilon^n t$, $n \geq 0$, while ε is the detuning parameter [35]

It results in the system

$$\ddot{z}_2 + \omega_2^2 z_2 = a_3 z_3^3 + a_2 z_3^2 z_2 + a_1 z_3 z_2^2 + b_1 z_3 + b_3 z_2^3 + c, \quad (27)$$

$$\ddot{z}_3 + \omega_3^2 z_3 = d_3 z_2^3 + d_2 z_2^2 z_3 + d_1 z_2 z_3^2 + e_1 z_2 + e_3 z_3^3 + f, \quad (28)$$

where the notations are obvious,

$$\omega_2^2 = \frac{8k_{21}}{m_2}, \quad a_3 = \frac{4k_{21}^*}{m_2}, \quad a_2 = -\frac{12k_{21}^*}{m_2}, \quad a_1 = \frac{12k_{21}^*}{m_2}, \quad b_3 = -\frac{8k_{21}^*}{m_2}, \quad b_1 = \frac{4k_{21}^*}{m_2}, \quad c = -g \quad (29)$$

$$\omega_3^2 = \frac{4k_{21}}{m_3 + m_4}, \quad d_3 = \frac{4k_{21}^*}{m_3 + m_4}, \quad d_2 = -\frac{12k_{21}^*}{m_3 + m_4}, \quad d_1 = \frac{12k_{21}^*}{m_3 + m_4}, \quad (30)$$

$$e_3 = -\frac{4k_{21}^*}{m_3 + m_4}, \quad e_1 = \frac{4k_{21}}{m_3 + m_4}, \quad f = -g$$

One gets

$$\frac{d^2 z_2}{dt^2} = \varepsilon \frac{\partial^2 z_{21}}{\partial T_0^2} + \varepsilon^3 \left(\frac{\partial^2 z_{23}}{\partial T_0^2} + 2 \frac{\partial^2 z_{21}}{\partial T_0 \partial T_2} \right) + \dots \quad (31)$$

$$\frac{d^2 z_3}{dt^2} = \varepsilon \frac{\partial^2 z_{31}}{\partial T_0^2} + \varepsilon^3 \left(\frac{\partial^2 z_{33}}{\partial T_0^2} + 2 \frac{\partial^2 z_{31}}{\partial T_0 \partial T_2} \right) + \dots \quad (32)$$

$$a_3 z_3^3 + a_2 z_3^2 z_2 + a_1 z_3 z_2^2 + b_3 z_2^3 + b_1 z_3 = \varepsilon b_1 z_{31} + \varepsilon^3 (a_3 z_{31}^3 + b_3 z_{21}^3 + b_1 z_{23}) + \dots, \quad (33)$$

$$d_3 z_2^3 + d_2 z_2^2 z_3 + d_1 z_2 z_3^2 + e_3 z_3^3 + e_1 z_2 = \varepsilon e_1 z_{21} + \varepsilon^3 (d_3 z_{21}^3 + e_3 z_{31}^3 + e_1 z_{23}) + \dots \quad (34)$$

One will assume that $c = \varepsilon^3 \bar{c}$ and $f = \varepsilon^3 \bar{f}$.

For the first power of ε one obtains the system

$$\frac{\partial^2 z_{21}}{\partial T_0^2} + \omega_2^2 z_{21} = b_1 z_{31}, \quad \frac{\partial^2 z_{31}}{\partial T_0^2} + \omega_3^2 z_{31} = e_1 z_{21} \quad (35)$$

From the first equation (35) it results

$$z_{31} = \frac{1}{b_1} \left(\frac{\partial^2 z_{21}}{\partial T_0^2} + \omega_2^2 z_{21} \right) \quad (36)$$

Replacing in the second relation (35), one gets

$$\frac{1}{b_1} \left(\frac{\partial^4 z_{21}}{\partial T_0^4} \right) + \frac{\omega_2^2 + \omega_3^2}{b_1} \frac{\partial^2 z_{21}}{\partial T_0^2} + \left(\frac{\omega_2^2 \omega_3^2}{b_1} - e_1 \right) z_{21} = 0. \quad (37)$$

One is looking for a solution in the form of $z_{21} = A_{21}(T_2, \dots) e^{i r_2 T_0}$. The last relation offers

$$\frac{1}{b_1} A_{21} r_2^4 + \frac{\omega_2^2 + \omega_3^2}{b_1} A_{21} r_2^2 + \left(\frac{\omega_2^2 \omega_3^2}{b_1} - e_1 \right) A_{21} = 0, \quad (38)$$

where from:

$$r_2^4 + (\omega_2^2 + \omega_3^2) r_2^2 + \omega_2^2 \omega_3^2 - b_1 e_1 = 0. \quad (39)$$

The last relation can be put in the form of a bi-square equation

$$r_2^4 + \alpha r_2^2 + \beta = 0 \quad (40)$$

where $\alpha > 0$, $\beta > 0$ and

$$\alpha^2 - 4\beta = (\omega_3^2 - \omega_2^2)^2 + 4b_1 e_1 > 0 \quad (41)$$

consequently, all the roots of the equation (40) are pure imaginary.

Analogically, one may prove a similar result for z_{31} . Moreover, the solutions of the two equations (35) are

$$z_{21} = A_{21}(T_2, \dots) (e^{i r_{21} T_0} + e^{i r_{22} T_0}) + cc \quad (42)$$

$$z_{31} = A_{31}(T_2, \dots) (e^{i r_{31} T_0} + e^{i r_{32} T_0}) + cc$$

where cc stays for complex conjugate.

Moreover, the roots of the equation (40) are distinctive and can be written as r_{21} , $-r_{21}$, r_{22} , and $-r_{22}$; analogically for the corresponding roots r_{31} ,

$-r_{31}$, r_{32} and $-r_{32}$. In conclusion, the expressions (42) hold true.

For the third power of ε one obtains the system

$$\frac{\partial^2 z_{23}}{\partial T_0^2} + \omega_2^2 z_{23} = -2 \frac{\partial^2 z_{21}}{\partial T_0 \partial T_2} + a_3 z_{31}^3 \quad (43)$$

$$+ a_2 z_{21} z_{31}^2 + a_1 z_{21} z_{31}^2 + b_3 z_{21}^3 + b_1 z_{33} + \bar{c},$$

$$\frac{\partial^2 z_{33}}{\partial T_0^2} + \omega_3^2 z_{33} = -2 \frac{\partial^2 z_{31}}{\partial T_0 \partial T_2} + d_3 z_{21}^3 + \quad (44)$$

$$d_2 z_{31} z_{21}^2 + d_1 z_{31}^2 z_{21} + e_3 z_{31}^3 + e_1 z_{23} + \bar{f},$$

with

$$\frac{\partial^2 z_{21}}{\partial T_0 \partial T_2} = \frac{\partial A_{21}}{\partial T_2} (i r_{21} e^{i r_{21} T_0} + i r_{22} e^{i r_{22} T_0}) + \frac{\partial A_{21}}{\partial T_2} (-i r_{21} e^{-i r_{21} T_0} - i r_{22} e^{-i r_{22} T_0}), \quad (45)$$

$$\frac{\partial^2 z_{31}}{\partial T_0 \partial T_2} = \frac{\partial A_{31}}{\partial T_2} (i r_{31} e^{i r_{31} T_0} + i r_{32} e^{i r_{32} T_0}) + \frac{\partial A_{31}}{\partial T_2} (-i r_{31} e^{-i r_{31} T_0} - i r_{32} e^{-i r_{32} T_0}). \quad (46)$$

One obtains the system

$$\frac{\partial^2 z_{23}}{\partial T_0^2} + \omega_2^2 z_{23} = -2 \left[\frac{\partial A_{21}}{\partial T_2} (i r_{21} e^{i r_{21} T_0} + i r_{22} e^{i r_{22} T_0}) + \frac{\partial A_{21}}{\partial T_2} (-i r_{21} e^{-i r_{21} T_0} - i r_{22} e^{-i r_{22} T_0}) \right] \quad (47)$$

$$+ a_3 z_{31}^3 + b_3 z_{21}^3 + b_1 z_{33} + \bar{c},$$

$$+ a_3 z_{31}^3 + b_3 z_{21}^3 + b_1 z_{33} + \bar{c},$$

$$\frac{\partial^2 z_{33}}{\partial T_0^2} + \omega_3^2 z_{33} = -2 \left[\frac{\partial A_{31}}{\partial T_2} (i r_{31} e^{i r_{31} T_0} + i r_{32} e^{i r_{32} T_0}) + \frac{\partial A_{31}}{\partial T_2} (-i r_{31} e^{-i r_{31} T_0} - i r_{32} e^{-i r_{32} T_0}) \right] \quad (48)$$

$$+ d_3 z_{21}^3 + e_3 z_{31}^3 + e_1 z_{23} + \bar{f}.$$

Case 2. This case is characterized by $z_1(t) \neq 0$, when the earthquake is active. The period of time for which this case is presented is a short one. It is assumed that the earthquake is defined by

$$z_1 = A_1 \cos(p_1 t + \varphi_1), \quad (49)$$

where A_1 is the amplitude of the earthquake, p_1 is the pulsation of the earthquake, while φ_1 is the initial phase of the earthquake. One may assume that

$$A_1 = \varepsilon \bar{A}_1 \quad (50)$$

The system (35) reads now

$$\begin{aligned} \frac{\partial^2 z_{21}}{\partial T_0^2} + \omega_2^2 z_{21} &= b_1 z_{31} + b_1 \bar{A}_1 \cos(p_1 t + \varphi_1) \\ \frac{\partial^2 z_{31}}{\partial T_0^2} + \omega_3^2 z_{31} &= e_1 z_{21} \end{aligned} \quad (51)$$

Following the same method of calculation, it successively results

$$z_{31} = \frac{1}{b_1} \left[\frac{\partial^2 z_{21}}{\partial T_0^2} + \omega_2^2 z_{21} - b_1 \bar{A}_1 \cos(p_1 t + \varphi_1) \right] \quad (52)$$

$$\begin{aligned} \frac{1}{b_1} \frac{\partial^4 z_{21}}{\partial T_0^4} + \frac{\omega_2^2 + \omega_3^2}{b_1} \frac{\partial^2 z_{21}}{\partial T_0^2} \\ + \left(\frac{\omega_2^2 \omega_3^2}{b_1} - e_1 \right) z_{21} \\ + \frac{(p_1^2 - 1)}{b_1} \bar{A}_1 \cos(p_1 t + \varphi_1) = 0. \end{aligned} \quad (53)$$

The solution is obtained as the sum between the solution of the homogeneous equation (it is the same as that given by the first relation (42)) and a particular solution of the non-homogeneous equation; this last solution is in the form of

$$z_{21nh} = B_2 \cos(p_1 t + \varphi_1) + C_2 \sin(p_1 t + \varphi_1) \quad (54)$$

where from

$$\begin{aligned} \frac{\partial z_{21nh}}{\partial T_0} = \frac{\partial z_{21nh}}{\partial t} &= -p_1 B_2 \sin(p_1 t + \varphi_1) \\ &+ p_1 C_2 \cos(p_1 t + \varphi_1), \\ \frac{\partial^2 z_{21nh}}{\partial T_0^2} = \frac{\partial^2 z_{21nh}}{\partial t^2} &= -p_1^2 B_2 \cos(p_1 t + \varphi_1) \\ &- p_1^2 C_2 \sin(p_1 t + \varphi_1), \\ \frac{\partial^4 z_{21nh}}{\partial T_0^4} = \frac{\partial^4 z_{21nh}}{\partial t^4} &= p_1^4 B_2 \cos(p_1 t + \varphi_1) \\ &+ p_1^4 C_2 \sin(p_1 t + \varphi_1). \end{aligned} \quad (55)$$

One obtains the equations:

- in sinus

$$\begin{aligned} \frac{1}{b_1} C_2 p_1^4 - \frac{\omega_2^2 + \omega_3^2}{b_1} C_2 p_1^2 \\ + \left(\frac{\omega_2^2 \omega_3^2}{b_1} - e_1 \right) C_2 = 0, \end{aligned} \quad (56)$$

where from either $C_2 = 0$ or a bi-square equation in p_1 (the condition for resonance). It results in that p_1

must not be equal to the values of resonance. It is easy to prove that equation (56) always has a root for which $p_1^2 > 0$, so the resonance may appear;

- in cosine

$$\begin{aligned} \frac{1}{b_1} B_2 p_1^4 - \frac{\omega_2^2 + \omega_3^2}{b_1} B_2 p_1^2 \\ + \left(\frac{\omega_2^2 \omega_3^2}{b_1} - e_1 \right) B_2 = (1 - p_1^2) \bar{A}_1, \end{aligned} \quad (57)$$

where one may determine the value of B_2 .

Further on, the calculations are similar to those presented for case 1

4. NUMERICAL RESULTS

For the numerical simulation, one will assume the following standard values: number of iterations $n_{iter} = 12000$; the increment of time $dt_{iter} = 10^{-3}$ [s]; the lengths $l_1 = 0.225$ [m], $l_2 = 0.2$ [m], $L_{12} = 0.45$ [m], $L_{23} = 0.45$ [m]; the masses $m_1 = 5$ [kg], $m_2 = 15$ [kg], $m_3 = 15$ [kg], $m_4 = 20$ [kg]; the elastic constants $k_{11} = 12.5 \times 10^3$ [N/m], $k_{12} = 12.5 \times 10^3$ [N/m], $k_{13} = 12.5 \times 10^3$ [N/m], $k_{14} = 12.5 \times 10^3$ [N/m], $k_{11}^* = 10^6$ [N/m³], $k_{12}^* = 10^6$ [N/m³], $k_{13}^* = 10^6$ [N/m³], $k_{14}^* = 10^6$ [N/m³], $k_{21} = 12.5 \times 10^3$ [N/m], $k_{22} = 12.5 \times 10^3$ [N/m], $k_{23} = 12.5 \times 10^3$ [N/m], $k_{24} = 12.5 \times 10^3$ [N/m], $k_{21}^* = 10^6$ [N/m³], $k_{22}^* = 10^6$ [N/m³], $k_{23}^* = 10^6$ [N/m³], $k_{24}^* = 10^6$ [N/m³], the excitations

$$z_1(t) = \begin{cases} z_1^0 \sin(\omega_{1z_1} t), & \text{for } 0 \leq t \leq 6[\text{s}], \\ 0, & \text{otherwise,} \end{cases} \quad (58)$$

$$\theta_1(t) = \begin{cases} \theta_1^0 \sin(\omega_{1\theta_1} t), & \text{for } 0 \leq t \leq 6[\text{s}], \\ 0, & \text{otherwise,} \end{cases} \quad (59)$$

$$\psi_1(t) = \begin{cases} \psi_1^0 \sin(\omega_{1\psi_1} t), & \text{for } 0 \leq t \leq 6[\text{s}], \\ 0, & \text{otherwise,} \end{cases} \quad (60)$$

where

$\omega_{1z_1} = \pi$ [rad/s], $\omega_{1\theta_1} = \pi$ [rad/s], $\omega_{1\psi_1} = \pi$ [rad/s];

the amplitude of the oscillations $z_1^0 = 0.25$ [m],

$\psi_1^0 = 0.001$ [rad], $\theta_1^0 = 0.001$ [rad]; the moments of

inertia $J_{x_2} = \frac{m_2 l_1^2}{12} = 0.06328125$ [kgm²],

$$J_{y_2} = \frac{m_2 l_2^2}{12} = 0.05 [\text{kgm}^2],$$

$$J_{x_3} = \frac{(m_3 + m_4) l_1^2}{12} = 0.14765625 [\text{kgm}^2],$$

$$J_{y_3} = \frac{(m_3 + m_4) l_2^2}{12} = 0.11666666 [\text{kgm}^2], \quad \text{the}$$

gravitational acceleration $g = 9.8065 [\text{m/s}^2]$ [1].

This is the standard situation.

In Table 1, some results are presented in a few situations described below. In the column named Parameters, only the values of the parameters that differ from the standard case are mentioned.

Table 1. Cases and results

Case	Description	Parameters	Results
1	Standard without earthquake	Standard	$T_{z_2} = 0.156 [\text{s}]$, $T_{z_3} = 0.164 [\text{s}]$, $\bar{z}_2 = -0.00558 [\text{m}]$, $\bar{z}_3 = -0.00981 [\text{m}]$
2	No earthquake, linear	Standard, $k_{ij}^* = 0$	$T_{z_2} = 0.236 [\text{s}]$, $T_{z_3} = 0.252 [\text{s}]$, $\bar{z}_2 = -0.01097 [\text{m}]$, $\bar{z}_3 = -0.01878 [\text{m}]$
3	No earthquake, linear	$k_{ij} = 25 \times 10^3$, $k_{ij}^* = 0$	$T_{z_2} = 0.048 [\text{s}]$, $T_{z_3} = 0.176 [\text{s}]$, $\bar{z}_2 = -0.00091 [\text{m}]$, $\bar{z}_3 = -0.00752 [\text{m}]$
4	No earthquake, linear	$k_{ij} = 50 \times 10^3$, $k_{ij}^* = 0$	$T_{z_2} = 0.124 [\text{s}]$, $T_{z_3} = 0.124 [\text{s}]$, $\bar{z}_2 = -0.00266 [\text{m}]$, $\bar{z}_3 = -0.00444 [\text{m}]$
5	No earthquake, nonlinear	Standard, $k_{ij}^* = 2 \times 10^6$	$T_{z_2} = 0.268 [\text{s}]$, $T_{z_3} = 0.244 [\text{s}]$, $\bar{z}_2 = -0.00976 [\text{m}]$, $\bar{z}_3 = -0.01714 [\text{m}]$
6	No earthquake, nonlinear	Standard, $k_{ij}^* = 5 \times 10^6$	$T_{z_2} = 0.228 [\text{s}]$, $T_{z_3} = 0.240 [\text{s}]$, $\bar{z}_2 = -0.01027 [\text{m}]$, $\bar{z}_3 = -0.01744 [\text{m}]$
7	Standard with earthquake	Standard	$T_{z_2} = 0.176 [\text{s}]$, $T_{z_3} = 0.200 [\text{s}]$, $\bar{z}_2 = 0.00326 [\text{m}]$, $\bar{z}_3 = -0.00244 [\text{m}]$

8	Earthquake, linear	Standard, $k_{ij}^* = 0$	$T_{z_2} = 0.052 [\text{s}]$, $T_{z_3} = 0.240 [\text{s}]$, $\bar{z}_2 = -0.01090 [\text{m}]$, $\bar{z}_3 = -0.01647 [\text{m}]$
9	Earthquake, linear	$k_{ij} = 25 \times 10^3$, $k_{ij}^* = 0$	$T_{z_2} = 0.044 [\text{s}]$, $T_{z_3} = 0.212 [\text{s}]$, $\bar{z}_2 = -0.00925 [\text{m}]$, $\bar{z}_3 = -0.00986 [\text{m}]$
10	Earthquake, linear	$k_{ij} = 50 \times 10^3$, $k_{ij}^* = 0$	$T_{z_2} = 0.064 [\text{s}]$, $T_{z_3} = 0.0520 [\text{s}]$, $\bar{z}_2 = 0.00458 [\text{m}]$, $\bar{z}_3 = 0.01218 [\text{m}]$
11	Earthquake, nonlinear	Standard, $k_{ij}^* = 2 \times 10^6$	$T_{z_2} = 0.024 [\text{s}]$, $T_{z_3} = 0.120 [\text{s}]$, $\bar{z}_2 = -0.00743 [\text{m}]$, $\bar{z}_3 = -0.01471 [\text{m}]$
12	Earthquake, nonlinear	Standard, $k_{ij}^* = 5 \times 10^6$	$T_{z_2} = 0.056 [\text{s}]$, $T_{z_3} = 0.024 [\text{s}]$, $\bar{z}_2 = -0.01157 [\text{m}]$, $\bar{z}_3 = -0.02681 [\text{m}]$

In the previous table T_{z_2} and T_{z_3} signifies the periods of the parameters z_2 and z_3 , respectively. Moreover \bar{z}_2 and \bar{z}_3 stands for the average values of z_2 and z_3 , respectively.

One can see that the periods in the standard case are shorter than those in the linear case; that is, the nonlinearities decrease the periods (compare case 1 to case 2). Moreover, the nonlinearities decrease the moduli of the average values (for both $|\bar{z}_2|$ and $|\bar{z}_3|$ (compare again case 1 to case 2). The linear stiffness has a complex influence, that is, on some intervals it decreases both the periods of vibrations and the average values of them, but on other intervals it increases some period and some average value, but it decreases other period and average value (compare case 2 to case 3 and case 4; it is clear that increasing the value of k_{ij} , T_{z_2} decreases at the beginning and then it increases, while T_{z_3} decreases for all interval of study; for $|\bar{z}_2|$ one may state that for the first part of interval of variation it decreases and then it increases, while for $|\bar{z}_3|$ its average value decreases all time). It looks like one can find a certain value for the linear stiffness for which some parameters (average value, period) may have extreme value.

Increasing the nonlinear stiffness (compare cases 5 and 6), the periods of oscillations and the moduli of the average amplitudes of vibrations increase or decrease, but with small corresponding values; it looks like there is a saturation of these increases or decreases, that is, there are some limiting values for them.

For the existence of an earthquake, the values are determined for the interval of time between 10 and 11 seconds, in order to diminish the contribution of the earthquake (this statement holds true for the cases without an earthquake, for better comparisons between different cases considered in the paper). The earthquake changes the vibration periods and shifts their average values (compare case 1 to case 7). Both values of the periods T_{z_2} and T_{z_3} increase in case 7 compared to case 1. The modulus of the average value of z_2 decreases, while the modulus of the average value of z_3 increases. It is possible to obtain positive values for at least one value of these average values. As the linear part of the stiffness increases, the periods first decrease, while the average values may shift from negative to positive (cases 8, 9, and 10). The nonlinear part of the stiffness reduces the vibration periods (within a certain range of variation) (compare case 7 to case 11 and case 12). It looks like there exists a point of minimum for each considered period. On the other hand, the moduli of the average values of z_2 and z_3 increase.

The linear cases (with or without earthquake) can also be studied analytically, and exact formulas can be obtained.

The physical explanation of these results is linked to the signs of the coefficients a_3 , b_3 , d_3 , and e_3 . In this paper a_3 and d_3 are positive, while b_3 and e_3 are negative. Moreover $|a_3| \neq |b_3|$, and $|d_3| = |e_3|$. More studies have to be conducted in order to highlight the influence of all parameters.

Some time histories are presented in Figs. 2, 3, 4, and 5. The time histories are drawn for different cases.

Analyzing the previous diagrams, one may state: i) after the end of the earthquake, the amplitudes of vibrations decrease; ii) after the end of the earthquake, the quasi-periods of the vibrations increase. The explanation is that after the end of the earthquake, the forced components of the vibrations vanish. One may see that during the earthquake, there exist a few components of the vibrations which superimpose over the forced components. These components remain after the earthquake vanishes, but their contributions to the final vibration are smaller.

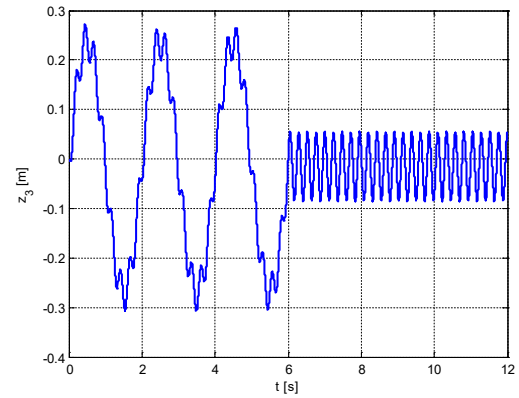


Figure 2. Time history $z_3 = z_3(t)$ in case 7

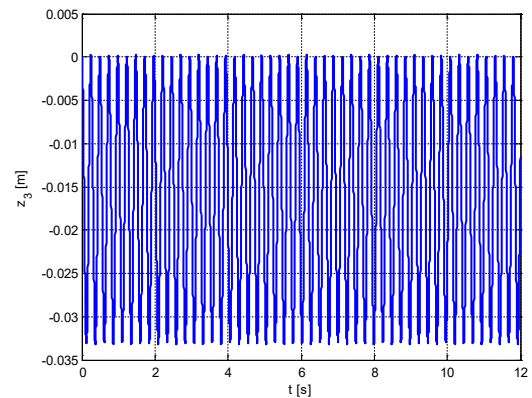


Figure 3. Time history $z_3 = z_3(t)$ in case 1

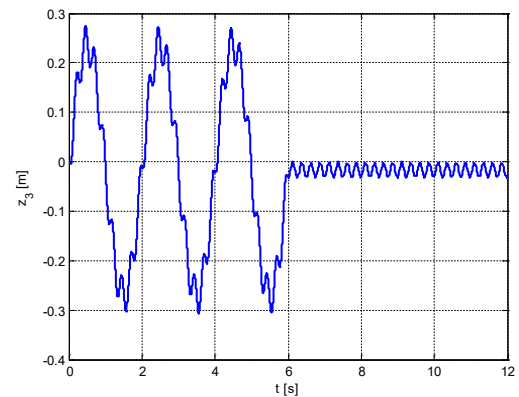


Figure 4. Time history $z_3 = z_3(t)$ in case 8

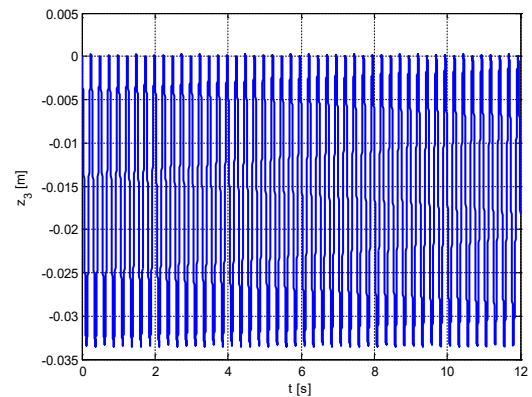


Figure 5. Time history $z_3 = z_3(t)$ in case 2.

In addition, one considers the case when all parameters are presented (z_1 , θ_1 , and ω_1 , that is, the

system has 6 degrees of freedom). The earthquake is also presented. For this case, the results are as follows: $T_{z_2} = 0.176$ [s], $\bar{z}_2 = 0.00326$ [m], $T_{z_3} = 0.200$ [s], $\bar{z}_3 = -0.00244$ [m], $T_{\psi_2} = 0.036$ [s], $\bar{\psi}_2 = 0.00001059$ [rad], $T_{\psi_3} = 0.072$ [s], $\bar{\psi}_3 = 0.00000117$ [rad], $T_{\theta_2} = 0.036$ [s], $\bar{\theta}_2 = 0.00001059$ [rad], $T_{\theta_3} = 0.052$ [s], $\bar{\theta}_3 = 0.00001055$ [rad]. It is easy to observe the influence of the earthquake: i) the average values cannot be predicted, the values being positive or negative; ii) the periods are very different even for oscillations that correspond to parallel axes (T_{z_2} and T_{z_3} , T_{ψ_2} and T_{ψ_3} , or T_{θ_2} and T_{θ_3}); iii) there exists no connection between the periods of motion; iv) there exists no connection between the average values.

The variation $z_3 = z_3(t)$ is captured in Fig. 6, and the time history $\theta_3 = \theta_3(t)$ is given in Fig. 7.

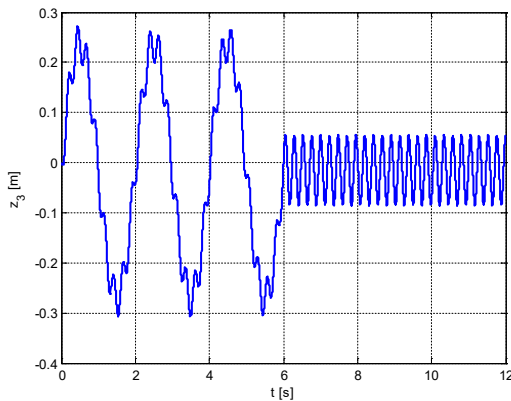


Figure 6. Time history $z_3 = z_3(t)$ in the case with all parameters presented.

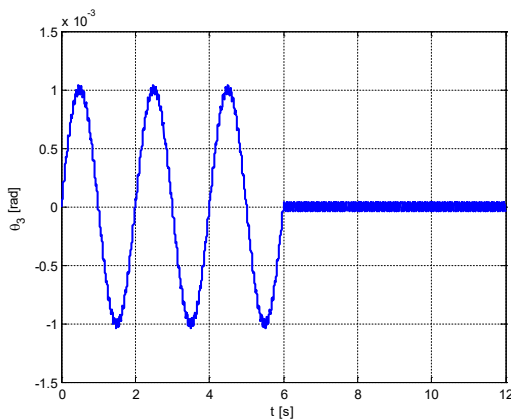


Figure 7. Time history $\theta_3 = \theta_3(t)$ in the case with all parameters presented

One may see that the rotational vibrations have small contributions to the vertical vibrations. This aspect justifies the simplified model (with only two linear degrees of freedom) used in this paper.

5. CONCLUSIONS

The existence of the nonlinearities changes the periods of vibrations and the average points of them. The situation is more dramatic when the earthquake is also present. In this situation, no matter which case is considered, the quasi-periods of the vibration changes (by increasing or decreasing their values; for the medical device, it is important that the T_{z_3} the quasi-period does not coincide with one of its eigenperiods. For instance, in the case of the existence of an earthquake, the quasi-period T_{z_2} reduces, while the average position \bar{z}_2 moves from a negative value to a positive value.

The analytical approach complicates the calculations very much. In this paper, only vibrations with two degrees of freedom are considered for the analytical approach, in two situations: with and without an earthquake. The presence of the earthquake modifies the expressions of different coefficients of the system of differential equations (some coefficients become functions of time). Direct integration is possible only for a linear system of differential equations. In all cases, with and without earthquake, linear or non-linear stiffness, the resonance may appear in the system. The numerical approach looks to be the usual way of integration in the most general situation.

The reader may observe that in the system presented in this paper, similar to a building, there is no vibration neutralizer (compare to Reference [20]). More studies have to be conducted from this point of view.

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