
Study on Natural Modes of Vibration For Clamped-Free Annular Plates

Laura-Mariana BARBU

Department of Mechanics, Faculty of Biotechnical Systems Engineering, National University of Science and Technology Politehnica Bucharest, Romania, laura_mariana.barbu@stud.aero.upb.ro

Roxana-Alexandra PETRE

Department of Mechanics, Faculty of Biotechnical Systems Engineering, National University of Science and Technology Politehnica Bucharest, Romania, roxana.petre@upb.ro

Cristian Mihail STĂNICĂ

INCDT COMOTI, Bucharest, Romania, e-mail: mihail.stanica@gmail.com

Mihai Valentin PREDOI*

Department of Mechanics, Faculty of Biotechnical Systems Engineering, National University of Science and Technology Politehnica Bucharest, Romania, mihai.predoi@upb.ro

* Author to whom correspondence should be addressed

Abstract: - This work presents an analytical and numerical investigation of the natural frequencies and mode shapes of clamped-free annular plates, widely used in engineering applications. The characteristic equations are derived for several numbers of nodal diameters ($n=0,1,2,3$) based on an original and consistent formulation of the governing equations and boundary conditions. The obtained results are compared with classical references, revealing and correcting several discrepancies. Recurrence formulas for Bessel functions are used to obtain original, compact, and validated analytical expressions. A novel formulation for $n=3$, unavailable in existing literature, is developed with full detail. The results are evaluated numerically and validated through Finite Element Analysis (FEA), showing excellent agreement. Extensive non-dimensional frequency parameters are provided, offering a reliable reference that is easy to use in practical applications.

Keywords: - annular plates vibration, analytic model, finite element analysis

1. INTRODUCTION

Annular plates (circular plates with a central hole) are fundamental structural elements widely encountered in engineering applications such as aerospace components, rotating machinery, and civil structures, where accurate prediction of vibration characteristics is essential. The theoretical investigation of plate vibrations dates back several decades, with the classical results systematically compiled in Leissa's NASA report [1], which remains a cornerstone reference for both circular and annular plates under various boundary conditions, even if some errors exist. Subsequent studies have extended these classical formulations. For instance, Ito and Crandall [2] derived characteristic equations for circular plates with free boundaries, while Irie et al. [3] analyzed annular plates with elastic boundary conditions, emphasizing the

influence of boundary flexibility on natural frequencies and mode shapes. Such approaches are particularly relevant for practical applications where boundary conditions deviate from idealized assumptions. Laura et al. [4] investigated annular plates with mixed boundary conditions using the Rayleigh–Ritz method, providing valuable insights into systems with non-uniform edge constraints. The influence of boundary conditions on vibration behavior has also been examined by Predoi et al. [5], highlighting configurations such as plates fixed at the center and free at the outer boundary. In rotating systems, additional effects arise, as shown by Predoi and Bugaru [6], where complementary bending moments significantly affect natural frequencies. Modern numerical techniques have also been employed, such as the Differential Quadrature Method (DQM) used by Wang and Wang [7], which offers efficient and accurate solutions compared to

traditional methods such as Finite Element Analysis (FEA).

The influence of axial forces on structural vibrations, studied by Ceaşu et al. [8], is also relevant for plate-type structures.

Extensions to Mindlin plate theory, which incorporate shear deformation and rotary inertia, were developed by Hosseini-Hashemi and Taher [9], providing accurate solutions for thick plates where classical thin plate theory is insufficient.

Mixed analytical–numerical approaches have also been proposed, such as the method introduced by Eftekhari and Jafari [10] for both free and forced vibrations. Nonlinear vibration effects have been investigated by Dragomirescu et al. [11], highlighting the complexity of real structural behavior.

Shi et al. [12], analyzed annular sector plates; however, their formulation of the generalized distributed shear force is incomplete, which may affect the accuracy of the results.

Further developments include studies on variable thickness plates [13], functionally graded materials [14], thermal effects [15], and internal supports [16], reflecting the increasing complexity of modern engineering applications. Damage effects [17] and the accuracy of numerical models compared to analytical solutions [18] have also been investigated. More recently, Wang and Liu [19] introduced a machine learning-based approach for predicting the natural frequencies of annular plates, bridging the gap between traditional analytical methods and emerging data-driven techniques.

The vibrations of annular plates with an eccentric inner hole were investigated by Chen et al. [20]. A preliminary FEA analysis, relevant to the present study, was conducted for a national conference by Barbu et al. [21].

The present work develops a comprehensive analytical and numerical framework for the study of annular plate vibrations. The governing differential equations are formulated with particular emphasis on the implementation of boundary conditions. Special attention is given to the deduction of the generalized shear force, which is not consistently defined in the literature and has a direct impact on the accuracy of the predicted vibration nodes. The determinants leading to the characteristic equations corresponding to different numbers n of nodal diameters are derived in detail, being an important theoretical contribution of the present work. A key contribution of this study is the derivation of the analytical form of the determinant for $n=3$ nodal diameters, thus extending the existing results available in dedicated literature. The analytical results are numerically evaluated and validated

through comparison with both existing references and our own finite element simulations.

2. THE LOVE-KIRCHHOFF PLATE MODEL IN POLAR COORDINATES

This section is based on the work presented in [22] by one of the authors. The displacement field associated with the harmonic vibrations of the circular/annular plate (u, v, w) , in cylindrical coordinates (r, θ, z) (Figure 1) can be expressed as:

$$\begin{aligned} u(r, \theta, z, t) &= U(r, \theta, z) \cos(\omega t - \varphi) \\ &= -z W_{,r} \cos(\omega t - \varphi), \\ v(r, \theta, z, t) &= V(r, \theta, z) \cos(\omega t - \varphi) \\ &= -\frac{z}{r} W_{,\theta} \cos(\omega t - \varphi), \\ w(r, \theta, z, t) &= W(r, \theta) \cos(\omega t - \varphi), \end{aligned} \quad (1)$$

where $W(r, \theta)$ represents the local normal displacement of the thin plate, ω is the angular frequency, t is time, and φ is the initial phase of the free vibration.

According to the Love–Kirchhoff thin-plate model under plane stress assumptions, the strain components in polar coordinates are given by:

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u}{\partial r} = -z W_{,rr}; \\ \varepsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} = -\frac{z}{r} \left(\frac{W_{,\theta\theta}}{r} + W_{,r} \right); \\ \gamma_{r\theta} &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\ &= -2 \frac{z}{r} \left(W_{,r\theta} - \frac{W_{,\theta}}{r} \right). \end{aligned} \quad (2)$$

Using these strain components, the corresponding stresses under a plane stress assumption are:

$$\begin{aligned} \sigma_{rr} &= \frac{-zE}{1-\nu^2} \left(W_{,rr} + \nu \left(\frac{W_{,\theta\theta}}{r^2} + \frac{W_{,r}}{r} \right) \right); \\ \sigma_{\theta\theta} &= \frac{-zE}{1-\nu^2} \left(\frac{W_{,\theta\theta}}{r^2} + \frac{W_{,r}}{r} + \nu W_{,rr} \right); \\ \tau_{r\theta} &= \frac{E}{2(1+\nu)} \gamma_{r\theta} = \frac{-zE}{1+\nu} \left(\frac{W_{,r\theta}}{r} - \frac{W_{,\theta}}{r^2} \right). \end{aligned} \quad (3)$$

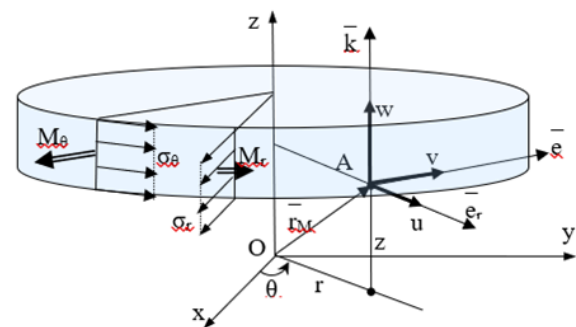


Figure 8. Displacement field (u, v, w) , stresses $(\sigma_{rr}, \sigma_{\theta\theta})$, and sectional bending moments (M_θ, M_r) [22]

The sectional bending moments $()$ required in the following developments are defined in terms of the

plate $D = \frac{Eh^3}{12(1-\nu^2)}$, where h denotes the plate thickness, as follows [22]:

$$M_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr} z dz = -D \left[W_{,rr} + \nu \left(\frac{W_{,\theta\theta}}{r^2} + \frac{W_{,r}}{r} \right) \right]; \quad (4)$$

$$M_{r\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{r\theta} z dz = -\frac{D(1-\nu)}{r} \left(W_{,r\theta} - \frac{W_{,\theta}}{r} \right)$$

The following generalized distributed sectional shear force is required to express the boundary conditions [22]:

$$\begin{aligned} V_r &= \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta} - D \frac{\partial}{\partial r} (\Delta W) \\ &= -\frac{D(1-\nu)}{r^2} \frac{\partial^2}{\partial \theta^2} \left(W_{,r} - \frac{W}{r} \right) - \\ &D \frac{\partial}{\partial r} \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) W \right] \end{aligned} \quad (5)$$

This generalized shear force includes a first term associated with the equilibrium of bending moments in the elementary volume. This contribution to V_r appears in all non-axially symmetric modes ($n > 0$) for which the sectional twisting moment $M_{r\theta}$ depends on the angular coordinate. However, this first term is often omitted in the literature, and only the incomplete form $V_r = -D \frac{\partial}{\partial r} (\Delta W)$ is retained, which may lead to inaccurate solutions.

The classical partial differential equation governing the free harmonic vibrations of plates in the Love-Kirchhoff model: $D\Delta\Delta W - \rho h \omega^2 W = 0$, can be written in polar coordinates [22] as follows:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 W(r, \theta) = k^4 W(r, \theta) \quad (6)$$

In this equation, $k^2 = \sqrt{\frac{\rho h}{D}} \omega = \sqrt{\frac{12\rho(1-\nu^2)}{Eh^2}} \omega$ is measured in $[m^{-2}]$. Considering only circumferentially periodic modal displacements:

$$W(r, \theta) = R_n(r) \cos n\theta; n = 0, 1, 2, \dots \quad (7)$$

Equation (6) can be factorized as follows:

$$M_r = -D[R_n''(r)] + \nu \left(\frac{R_n'(r)}{r} - \frac{n^2 R_n(r)}{r^2} \right) = -Dk^2 \left[R_n''(z) + \nu \left(\frac{R_n'(z)}{z} - \frac{n^2 R_n(z)}{z^2} \right) \right] \quad (12)$$

The boundary condition corresponding to the vanishing of the sectional bending moment M_r at $z=ka$, using the notation $\alpha=ka$, becomes:

$$\begin{cases} r^2 \frac{d^2 R_n}{dr^2} + r \frac{dR_n}{dr} + (n^2 - k^2 r^2) R_n = 0 \\ r^2 \frac{d^2 R_n}{dr^2} + r \frac{dR_n}{dr} - (n^2 + k^2 r^2) R_n = 0 \end{cases} \quad (8)$$

A change of variable $z=kr$, transforms the above differential equations into two canonical Bessel equations of integer order n :

$$\begin{cases} z^2 \frac{d^2 R_n}{dz^2} + z \frac{dR_n}{dz} + (n^2 - z^2) R_n(z) = 0 \\ z^2 \frac{d^2 R_n}{dz^2} + z \frac{dR_n}{dz} - (n^2 + z^2) R_n(z) = 0 \end{cases} \quad (9)$$

The general solution of the initial problem (6) is obtained as the sum of the general solutions of the system (9) of differential equations [22]:

$$R_n(z) = A_n J_n(z) + B_n Y_n(z) + C_n I_n(z) + D_n K_n(z) \quad (10)$$

where J_n and Y_n are Bessel functions of the first and second kinds, respectively, and I_n and K_n are the modified Bessel functions of the first and second kinds, respectively.

3. FREE VIBRATIONS OF ANNULAR PLATES. ANALYTICAL SOLUTIONS

The most relevant case in practical applications of annular plates, defined by their outer radius a , inner radius b , and thickness h , is the clamped-free configuration. In this case, the displacement and the radial slope vanish along the inner boundary: $W(b, \theta) = 0$; $W'(b, \theta) = 0 \quad \forall \theta \in [0, 2\pi]$. These boundary conditions, applied to the general solution (10), with the notation $\beta=kb$, lead to:

$$\begin{aligned} A_n J_n(\beta) + B_n Y_n(\beta) + C_n I_n(\beta) \\ + D_n K_n(\beta) = 0 \\ A_n J_n'(\beta) + B_n Y_n'(\beta) + C_n I_n'(\beta) \\ + D_n K_n'(\beta) = 0 \end{aligned} \quad (11)$$

For the free outer boundary $r=a$, the sectional bending moment and the sectional generalized shear force must vanish: $M_r(a, \theta) = 0$; $V_r(a, \theta) = 0$; $\forall \theta \in [0, 2\pi]$. Using eq. (4), the expression of the sectional bending moment becomes:

$$A_n \left[J_n''(\alpha) + \frac{\nu}{\alpha} J_n'(\alpha) - \frac{\nu n^2}{\alpha^2} J_n(\alpha) \right] + B_n \left[Y_n''(\alpha) + \frac{\nu}{\alpha} Y_n'(\alpha) - \frac{\nu n^2}{\alpha^2} Y_n(\alpha) \right] \\ + C_n \left[I_n''(\alpha) + \frac{\nu}{\alpha} I_n'(\alpha) - \frac{\nu n^2}{\alpha^2} I_n(\alpha) \right] + D_n \left[K_n''(\alpha) + \frac{\nu}{\alpha} K_n'(\alpha) - \frac{\nu n^2}{\alpha^2} K_n(\alpha) \right] = 0 \quad (13)$$

From eq. 5, the generalized sectional shear force can be expressed as:

$$V_r = \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta} - D \frac{\partial}{\partial r} (\Delta W) = \left\{ \frac{D(1-\nu)n^2}{r^2} \left(W_{,rr} - \frac{W}{r} \right) - D \frac{\partial}{\partial r} \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) W \right] \right\} \cos n\theta \quad (14)$$

Imposing the condition that the generalized shear force vanishes at $\alpha=ka$, for any angle θ , leads to:

$$\frac{(1-\nu)n^2}{r^2} \left(W' - \frac{W}{r} \right) - \left[W''' + \frac{1}{r} W'' - \frac{1}{r^2} W' - \frac{n^2}{r^2} W' + \frac{2n^2}{r^3} W \right] = 0 \quad (15)$$

The above condition can be separated into two terms: the first is associated with the variation of the edge sectional twisting moment $M_{r\theta}$, while the second term is commonly retained in literature. In order to evaluate the influence of neglecting the first term, the boundary condition can be written by retaining only the second contribution:

$$R_n'''(\alpha) + \frac{1}{\alpha} R_n''(\alpha) - \frac{1+n^2}{\alpha^2} R_n'(\alpha) + \frac{2n^2}{\alpha^3} R_n(\alpha) = 0 \quad (16)$$

However, the correct boundary condition for the generalized shear force is given by:

$$\begin{vmatrix} J_0(\beta) & Y_0(\beta) & I_0(\beta) & K_0(\beta) \\ J_1(\beta) & Y_1(\beta) & -I_1(\beta) & K_1(\beta) \\ J_0(\alpha) & Y_0(\alpha) & -I_0(\alpha) + 2BI_1(\alpha) & -K_0(\alpha) - 2BK_1(\alpha) \\ J_1(\alpha) & Y_1(\alpha) & I_1(\alpha) & -K_1(\alpha) \end{vmatrix} = 0 \quad (18)$$

The notation $B = \frac{1-\nu}{\alpha}$ was used in expressing this determinant. This result coincides with the characteristic determinant given in eq. 2.2.7 of ref. [1] for $n=0$, apart from row permutations and notational differences (e.g. λ instead of our α). This agreement is independent of the expression used for

$$R_n'''(\alpha) + \frac{1}{\alpha} R_n''(\alpha) - \frac{1+(2-\nu)n^2}{\alpha^2} R_n'(\alpha) + \frac{(3-\nu)n^2}{\alpha^3} R_n(\alpha) = 0 \quad (17)$$

The homogeneous algebraic system associated with the boundary conditions must cancel for a non-trivial solution ka or kb . Thus, the characteristic equation corresponding to each number n of nodal diameters is obtained by imposing that the associated determinant is equal to zero. We developed the above computations and obtained, for $n=0$:

the generalized shear force, since for $n=0$ the formulations (16) and (17) coincide.

For $n=1$ nodal diameters, our solution, which is validated by the numerical results presented in the next section, exhibits a change in sign in the last term of the second row, compared to ref. [1]:

$$\begin{vmatrix} J_1(\beta) & Y_1(\beta) & I_1(\beta) & K_1(\beta) \\ J_0(\beta) & Y_0(\beta) & I_0(\beta) & -K_0(\beta) \\ J_1(\alpha) & Y_1(\alpha) & -I_1(\alpha) + 2BI_2 & -K_1(\alpha) - 2BK_2(\alpha) \\ J_2(\alpha) & Y_2(\alpha) & I_2(\alpha) & -K_2(\alpha) \end{vmatrix} = 0 \quad (19)$$

The last two rows differ from the corresponding ones in ref. [1], but since they are shorter and yield equivalent results, we adopt the present formulation.

For $n=2$, attempts were made to retrieve the determinant indicated in ref. [1], but to no avail. We

obtain instead the following characteristic equation, in a compact form, using as notation $A = \frac{2(1-\nu)}{\alpha^2}$ and $B = \frac{1-\nu}{\alpha}$:

$$\begin{vmatrix} J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) \\ J_1(\beta) & Y_1(\beta) & I_1(\beta) & -K_1(\beta) \\ (A-1)J_2(\alpha) + BJ_3(\alpha) & (A-1)Y_2(\alpha) + BY_3(\alpha) & (A+1)I_2(\alpha) - BI_3(\alpha) & (A+1)K_2(\alpha) + BK_3(\alpha) \\ J_3(\alpha) - AJ_1(\alpha) & Y_3(\alpha) - AY_1(\alpha) & I_3(\alpha) - AI_1(\alpha) & -K_3(\alpha) - AK_1(\alpha) \end{vmatrix} = 0. \quad (20)$$

We extend the analytical formulation to the case of $n=3$ nodal diameters, for which, to the best of the

$$\begin{vmatrix} J_3(\beta) & Y_3(\beta) & I_3(\beta) & K_3(\beta) \\ J_2(\beta) & Y_2(\beta) & I_2(\beta) & -K_2(\beta) \\ (3A-1)J_3(\alpha) + BJ_4(\alpha) & (3A-1)Y_3(\alpha) + BY_4(\alpha) & (3A+1)I_3(\alpha) - BI_4(\alpha) & (3A+1)K_3(\alpha) + BK_4(\alpha) \\ CJ_4(\alpha) - \frac{6}{\alpha}J_3(\alpha) & CY_4(\alpha) - \frac{6}{\alpha}Y_3(\alpha) & DI_4(\alpha) + \frac{6}{\alpha}I_3(\alpha) & -DK_4(\alpha) + \frac{6}{\alpha}K_3(\alpha) \end{vmatrix} = 0 \quad (21)$$

with the same constants A , B and introducing parameters: $C = 1 + \frac{3}{\alpha}B + \frac{9A}{2}$ and $D = 1 - \frac{3}{\alpha}B - \frac{9A}{2}$.

In order to provide a general solution, applicable to a wider class of plates, the natural angular frequencies are presented in non-dimensional form using the classical notation: $\lambda^2 = k^2 a^2 = \sqrt{\frac{12\rho(1-\nu^2)}{Eh^2}} \omega a^2$ and $\gamma = \frac{b}{a}$. It should be noted that in ref. [1]: ρ denotes the area density of the material [kg/m^2], whereas in this work it represents the mass density of the material. It should also be noted that, by using the parameter λ^2 , one can easily determine the effective angular frequency in rad/s by specifying only the material parameters, the external radius a , and the plate thickness h , for the corresponding ratio b/a .

4. NUMERICAL STUDY

This section is divided into two subsections. Section 4.1 presents the non-dimensional frequency parameters obtained by numerically solving the analytical characteristic equations for various nodal diameters, while Section 4.2 gives the corresponding results obtained by FEA conducted in COMSOL, thus enabling a direct comparison between the analytical and numerical approaches.

4.1 Frequency parameters from numerical solutions of characteristic equations

The characteristic equations (18) for $n=0$, (19) for $n=1$, (20) for $n=2$, and (21) for $n=3$ were solved using a numerical application developed by the authors in MATLAB [23] for the considered range of $\gamma \in [0.1, 0.9]$ by a 0.1 step. The obtained non-dimensional frequency parameters are presented in Table 1. The accuracy of the roots is proven by plotting the absolute errors in the determined λ values (Figure 2) indicating an accuracy of $\pm 10^{-10}$, which is obviously even better in λ^2 .

On the other hand, it is important to evaluate the accuracy of the determined λ values (Figure 3) using the bisection method, as the interval of root location. The base 10 logarithm of this interval is used for readability, providing an accuracy of 10^{-6} to 10^{-14} .

author's knowledge, no corresponding analytical results have been reported in the literature:

We selected the $n=3$ case, as being more challenging for these plots.

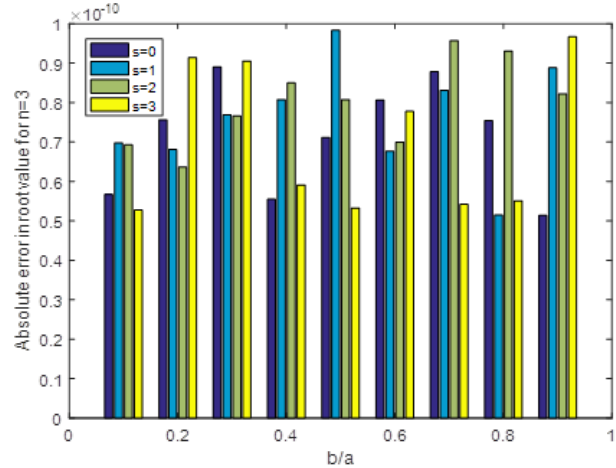


Figure 9. Absolute errors of the function at the determined roots

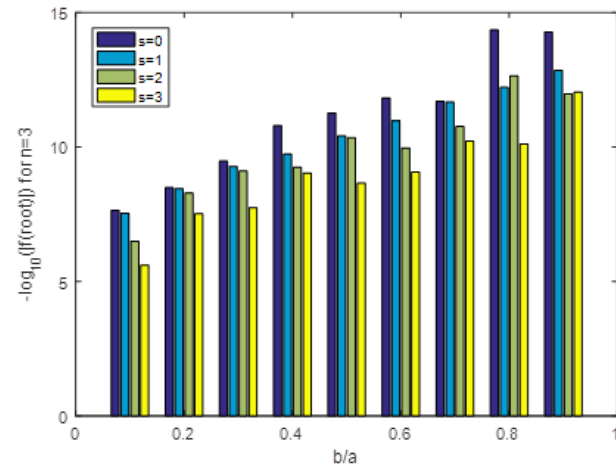


Figure 10. Errors in root location for the determined λ values on a logarithmic scale.

The Finite Element Analysis (FEA) was performed using the COMSOL software package (COMSOL Multiphysics User Manual). The finite element formulation used for plates is based on the Mindlin plate model, which accounts for the shear stress on the plate's cross-section in a more advanced manner than the classical Love-Kirchhoff model.

However, the two models converge for very thin plates ($h/a \leq 0.02$), and for this reason all FEA models used in this work satisfy $h/a < 0.02$. The outer radius of the model is $a=1$ m, and the corresponding $\gamma=b/a$ ratios are those specified above.

The material is steel, with $E=210$ GPa, $\nu=1/3$, and $\rho=7800$ kg/m³.

Since a large number of eigenmodes (>80) is required, the generated mesh contains 34,900 elements for $b/a=0.2$ and 12,800 elements for $b/a=0.8$ (Figure 4).

After post-processing, the frequency coefficients λ^2 are determined and included in Table 1. In this table, we have highlighted the incorrect values reported in the classical ref. [1], which are most likely due to typographical errors.

Our numerical solutions obtained from the analytical development of the characteristic equations are denoted as “Analytic”, whereas the corresponding numerical solutions obtained using the finite elements are denoted by “FEA”.

The first four natural vibration modes are shown in Figure 7 for $\gamma = b/a = 0.5$, corresponding to n

$=0,1,2$ and 3 nodal diameters. As the ratio b/a increases, the natural frequencies become very close to one another, making it difficult to identify the correct order n and

the number of circular nodes s . In the author’s opinion, this difficulty explains some of the erroneous values reported in ref. [1], which are corrected by the present results and validated through two distinct methods.

One important difficulty in using FEA modal analysis is the multiplicity of eigenmodes due to the geometrical symmetry of the structure. The user must use mode shape plots to correctly identify the mode order. We used a logical condition in separate plots in which positive and negative displacements are given uniform but distinct colors.

Table 5. Frequency parameters λ^2 for the clamped - free annular plate (n =no. nodal diameters; s =no. circular nodes) of $\gamma = b/a = 0.1 \dots 0.9$.

n	s	Formula	λ^2 for $\gamma = b/a$:								
			0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	[1]. tab. 2.30	4.23	-	6.66	-	13.0	-	37.0	-	51.5
		Analytic	4.2629	5.213	6.701	9.0719	13.089	20.607	37.069	84.678	344.76
		FEA	4.188	5.121	6.5828	8.916	12.873	20.284	36.527	83.515	340.38
1	0	[1]. tab. 2.30	3.14	-	6.33	-	13.3	-	37.5	-	345
		Analytic	3.4799	4.8171	6.5601	9.1293	13.313	20.976	37.567	85.289	345.47
		FEA	3.438	4.757	6.476	9.0094	13.135	20.694	37.066	84.168	341.13
2	0	[1]. tab. 2.30	5.62	-	7.96	-	14.7	-	39.3	-	347
		Analytic	5.5237	6.3689	7.9042	10.438	14.704	22.530	39.345	87.331	347.79
		FEA	5.557	6.372	7.8637	10.343	14.533	22.24	38.823	86.174	343.40
3	0	[1]. tab. 2.30	12.4	-	13.27	-	18.5	-	42.6	-	352
		Analytic	12.233	12.395	13.050	14.725	18.310	25.714	42.383	90.470	351.22
		FEA	12.306	12.468	13.121	14.789	18.345	25.677	42.16	89.655	347.20
0	1	[1]. tab. 2.30	25.3	-	42.6	-	85.1	-	239	-	970
		Analytic	25.346	32.386	42.721	58.673	85.179	134.10	240.08	543.83	2189.52
		FEA	24.973	31.926	42.135	57.9	84.111	132.52	237.57	537.30	2164.7
1	1	[1]. tab. 2.30	27.3	-	44.6	-	86.7	-	241	-	2189
		Analytic	27.725	34.588	44.706	60.469	86.821	135.61	241.49	545.15	2190.77
		FEA	27.357	34.134	44.128	59.706	85.763	134.04	238.99	538.64	2165.9
2	1	[1]. tab. 2.30	37.0	-	50.9	-	91.7	-	246	-	2194
		Analytic	36.922	41.944	50.939	65.951	91.763	140.14	245.70	549.10	2194.50
		FEA	36.522	41.848	50.372	65.211	90.737	138.62	243.24	542.63	2169.7
3	1	[1]. tab. 2.30	53.2	-	62.1	-	100	-	253	-	2200
		Analytic	53.106	55.310	61.949	75.343	100.07	147.70	252.71	555.68	2200.72
		FEA	52.607	54.787	61.352	74.604	99.068	146.21	250.3	549.27	2176.0

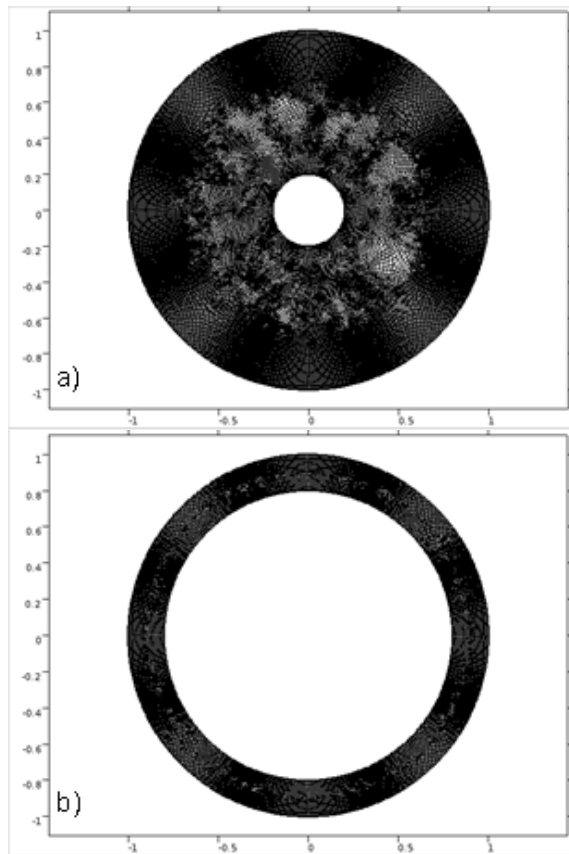


Figure 11. FEM mesh used for $b/a=0.2$ (a) and for $b/a=0.8$ (b)

The number of nodal diameters and circles is thus clearly visible. E.g. in Figure 8 two FEA vibration modes indicating the same natural frequency, clearly indicate the same mode with $n=3$ nodal diameters and $s=1$ nodal circle. This algorithm requires user intervention, which is time-consuming and thus motivates the use of our tabulated results.

The relative errors of FEA solutions vs. the analytical ones, from Table 1, are shown on Figure 5 for $n=3$, considered as the most challenging case. The maximum relative error is less than 2%,

attributable to the FEA element type used in COMSOL.

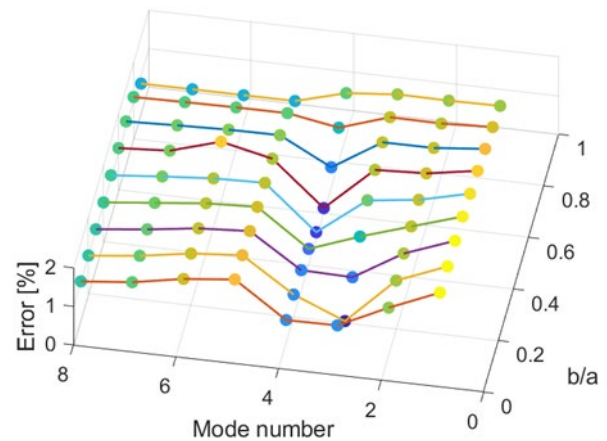


Figure 12. Relative errors comparison: FEM vs. analytic

As for the FEA eigenvalues accuracy, a convergence test was performed, progressively reducing the size of the finite elements used in the mesh. On Figure 6 are plotted for $b/a=0.9$, which is the most challenging case, the first four λ values for $s=0$ (Figure 6a) and for $s=a$ (Figure 6b) as functions of element size. As the size reduces to 5 mm, the obtained values remain in a 2% accepted tolerance for all λ values.

Other important remarks of the current study are the following:

- As the ratio $\gamma = b/a$ increases, the natural frequencies also increase, with a particularly sharp rise for $\gamma = 0.9$ requiring a high numerical accuracy. For this reason, ref. [1] reports incorrect results in several cases.
- For $\gamma < 0.4$, the natural frequency corresponding to $n = 1$ is lower than that corresponding to $n = 0$, indicating a lower bending energy in the case $n = 1$.

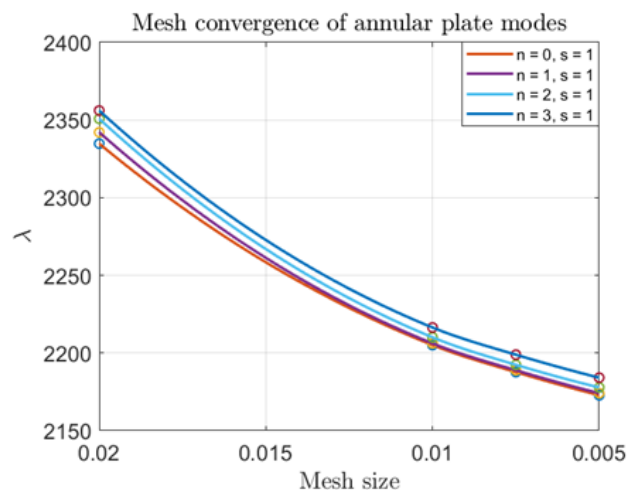
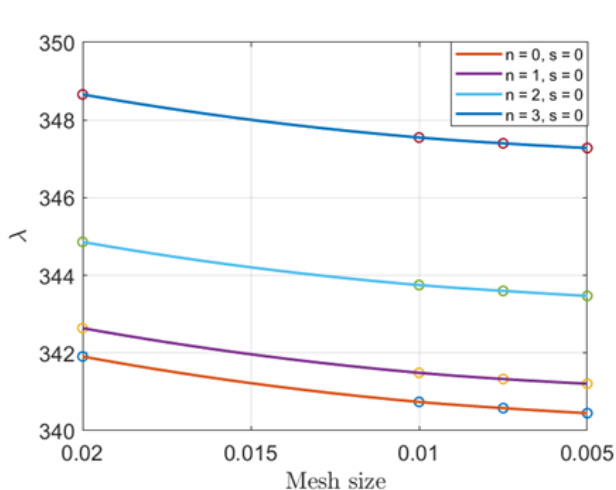


Figure 13. Eigenvalue convergence with element size (in meters) used in FEA mesh for $b/a=0.9$, and $s=0$ (a), $s=1$ (b)

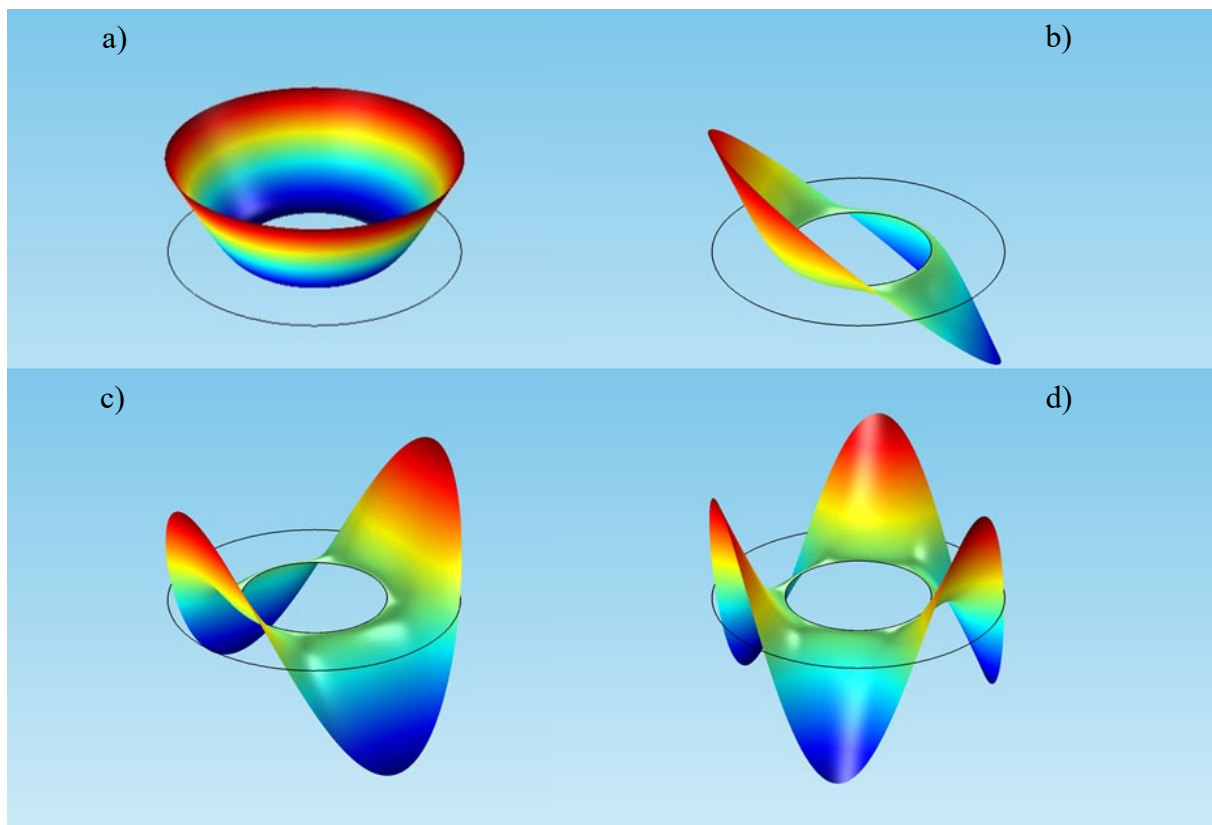


Figure 14. First four natural modes of vibration for $\gamma=0.5$: $(n=0,s=0)$ (a), $(n=1,s=0)$ (b), $(n=2,s=0)$ (c), $(n=3,s=0)$ (d).

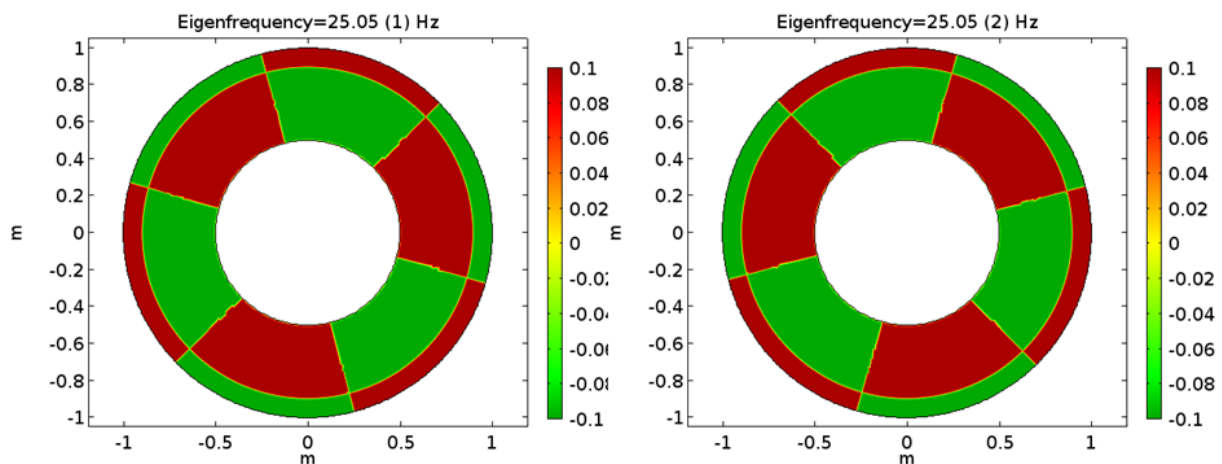


Figure 15. Multiple natural modes of vibration for $\gamma=0.5$: $(n=3,s=1)$ can be easily identified with such special plots

5. CONCLUSIONS

The natural frequencies and mode shapes of clamped-free annular plates are important in engineering applications such as rotating disc tools (grinders blades, disc cutters, etc.). The characteristic equations were deduced and verified for $n=0,1,2,3$ revealing some significant discrepancies in the available literature even if most numerical solutions correspond with ref. [1] to the given number of digits. Several values reported in literature were found to be inaccurate and are corrected by the present results. The characteristic

determinants were deduced in full detail, being validated through comparisons and corrections of existing expressions where available, and especially through our FEM simulations, all representing important contribution to existing theoretical aspects. Furthermore, the authors derived the determinant for the given boundary conditions and the associate characteristic equation for $n=3$ nodal diameters, which, to the best of the author's knowledge, is not available in the existing references. Probably in ref. [1] was considered to be sufficient a study for $n=0,1$ or 2 , but we consider that for modern high speed machines, order $n=3$ being clearly more complex than the lower order

ones and leading to considerably higher frequencies, is also important.

The numerical solutions λ^2 , obtained using our algorithm for solving the characteristic equations, exhibit high accuracy, with a maximum error of 10^{-12} in the function value.

The FEM simulations, based on the author's own model, provide a large set of eigenvalues, most of which appear as double solutions due to geometric symmetry. Consequently, a considerable post-processing effort is required to correctly identify the vibration modes. For this reason, the numerically determined angular frequencies, based on the non-dimensional angular frequencies λ^2 are considered to provide a reliable approach for practical applications.

With practical applications in mind, Appendix 2 provides tabulated non-dimensional angular frequencies for $n=0,1,2,3$ and $s=0,1,2,3$, as well as for Poisson's ratio $\nu=0.25; 0.3$ and 0.35 . The resulting values are not significantly affected by Poisson's ratio; however, for increased accuracy, all corresponding values are provided.

By developing the theoretical formulation leading to the numerical solution of the analytical characteristic equations, and by validating these results through FEM, the present work provides a reliable reference for clamped-free annular plates.

Appendix 1. Recurrence formulas for Bessel functions.

Recurrence formulas for Bessel functions of the first kind J_n and of the second kind Y_n , of non-negative integer order $n \geq 0$:

$$J_{n+1}(z) = \frac{2n}{z} J_n(z) - J_{n-1}(z);$$

$$Y_{n+1}(z) = \frac{2n}{z} Y_n(z) - Y_{n-1}(z)$$

Recurrence formulas for the modified Bessel functions of the first kind I_n and of the second kind K_n , of non-negative integer order n :

$$I_{n+1}(z) = I_{n-1}(z) - \frac{2n}{z} I_n(z);$$

$$K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z)$$

Derivatives of the Bessel functions used in the formulation of the boundary conditions for plate vibrations [25]:

$$\begin{aligned} J'_n(z) &= \frac{1}{2} [J_{n-1}(z) - J_{n+1}(z)] \\ &= \frac{n}{z} J_n(z) - J_{n+1}(z) = J_{n-1}(z) - \frac{n}{z} J_n(z) \end{aligned}$$

$$\begin{aligned} Y'_n(z) &= \frac{1}{2} [Y_{n-1}(z) - Y_{n+1}(z)] \\ &= \frac{n}{z} Y_n(z) - Y_{n+1}(z) \\ &= Y_{n-1}(z) - \frac{n}{z} Y_n(z) \end{aligned}$$

$$\begin{aligned} I'_n(z) &= \frac{1}{2} [I_{n-1}(z) + I_{n+1}(z)] = I_{n+1}(z) + \frac{n}{z} I_n(z) \\ &= I_{n-1}(z) - \frac{n}{z} I_n(z) \end{aligned}$$

$$\begin{aligned} K'_n(z) &= -\frac{1}{2} [K_{n-1}(z) + K_{n+1}(z)] \\ &= -K_{n+1}(z) + \frac{n}{z} K_n(z) \\ &= -K_{n-1}(z) - \frac{n}{z} K_n(z) \end{aligned}$$

The second derivatives are obtained as follows:

$$\begin{aligned} J''_n(z) &= \frac{d}{dz} \left[\frac{n}{z} J_n(z) - J_{n+1}(z) \right] \\ &= -\frac{n}{z^2} J_n(z) + \frac{n}{z} J'_n(z) - J'_{n+1}(z) \\ &= -\frac{n}{z^2} J_n(z) + \frac{n}{z} \left[\frac{n}{z} J_n(z) - J_{n+1}(z) \right] - J_{n+1}(z) \\ &\quad + \frac{n+1}{z} J_{n+1}(z) \\ &= J_n \left[\frac{n(n-1)}{z^2} - 1 \right] + \frac{J_{n+1}}{z} \end{aligned}$$

Similarly: $Y''_n(z) = Y_n \left[\frac{n(n-1)}{z^2} - 1 \right] + \frac{Y_{n+1}}{z}$. As for the modified Bessel functions:

$$\begin{aligned} I''_n(z) &= I_n \left[\frac{n(n-1)}{z^2} + 1 \right] - \frac{I_{n+1}}{z}; K''_n(z) \\ &= K_n \left[\frac{n(n-1)}{z^2} + 1 \right] + \frac{K_{n+1}}{z} \end{aligned}$$

The third derivatives were derived and used in the theoretical section of the manuscript:

$$\begin{aligned} J'''_n(z) &= \frac{n-1}{z} \left[\frac{n(n-2)}{z^2} - 1 \right] J_n \\ &\quad - \left(\frac{n^2+2}{z^2} - 1 \right) J_{n+1} \\ Y'''_n(z) &= \frac{n-1}{z} \left[\frac{n(n-2)}{z^2} - 1 \right] Y_n \\ &\quad - \left(\frac{n^2+2}{z^2} - 1 \right) Y_{n+1} \\ I'''_n(z) &= \frac{n-1}{z} \left[\frac{n(n-2)}{z^2} + 1 \right] I_n(z) \\ &\quad + \left(\frac{n^2+2}{z^2} + 1 \right) I_{n+1}(z) \\ K'''_n(z) &= \frac{n-1}{z} \left[\frac{n(n-2)}{z^2} + 1 \right] K_n(z) \\ &\quad - \left(\frac{n^2+2}{z^2} + 1 \right) K_{n+1}(z) \end{aligned}$$

Appendix 2. Non-dimensional angular frequency parameters

frequency parameter $\lambda^2 = k^2 a^2 = \sqrt{\frac{12\rho(1-\nu^2)}{Eh^2}} \omega a^2$
for annular plates, for $\nu = 0.25; 0.3; 0.35$.

The appendix contains tables of numerically computed values of the non-dimensional angular

Table 6. $n=0$ nodal diameters & $\nu = 0.25$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$s = 0$	4.1974	5.1306	6.5971	8.9414	12.924	20.392	36.776	84.231	343.86
$s = 1$	25.132	32.147	42.452	58.362	84.811	133.64	239.48	542.94	2187.77
$s = 2$	73.783	93.953	123.32	168.52	243.49	381.61	680.31	1534.75	6154.58
$s = 3$	146.58	186.30	244.04	332.90	480.25	751.58	1338.10	3014.82	12074.99

Table 7. $n=0$ nodal diameters & $\nu = 0.30$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$s = 0$	4.2374	5.1811	6.6604	9.0206	13.024	20.522	36.953	84.500	344.40
$s = 1$	25.262	32.291	42.614	58.549	85.033	133.91	239.84	543.48	2188.82
$s = 2$	73.901	94.084	123.47	168.69	243.69	381.86	680.65	1535.26	6155.58
$s = 3$	146.69	186.43	244.19	333.07	480.45	751.84	1338.43	3015.33	12076.00

Table 8. $n=0$ nodal diameters & $\nu = 0.35$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$s = 0$	4.2754	5.2294	6.7212	9.0971	13.122	20.649	37.127	84.766	344.94
$s = 1$	25.388	32.432	42.774	58.734	85.252	134.19	240.20	544.01	2189.87
$s = 2$	74.017	94.214	123.61	168.86	243.90	382.12	680.99	1535.76	6156.59
$s = 3$	146.81	186.56	244.34	333.24	480.66	752.09	1338.77	3015.83	12077.00

Table 9. $n=1$ nodal diameters & $\nu = 0.25$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$s = 0$	3.4754	4.8068	6.5403	9.0943	13.254	20.876	37.395	84.969	344.70
$s = 1$	27.593	34.431	44.518	60.239	86.532	135.23	240.97	544.34	2189.09
$s = 2$	77.396	96.849	125.70	170.54	245.26	383.18	681.74	1536.05	6155.78
$s = 3$	151.17	189.65	246.66	335.05	482.08	753.19	1339.52	3016.11	12076.17

Table 10. $n=1$ nodal diameters & $\nu = 0.30$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$s = 0$	3.4781	4.8130	6.5523	9.1155	13.290	20.936	37.498	85.162	345.17
$s = 1$	27.673	34.526	44.631	60.378	86.706	135.46	241.28	544.83	2190.10
$s = 2$	77.487	96.954	125.82	170.69	245.44	383.41	682.05	1536.53	6156.76
$s = 3$	151.27	189.76	246.79	335.20	482.27	753.42	1339.84	3016.60	12077.16

Table 11. $n=1$ nodal diameters & $\nu = 0.35$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$s = 0$	3.4808	4.8191	6.5640	9.1362	13.325	20.995	37.600	85.353	345.63
$s = 1$	27.752	34.619	44.743	60.515	86.878	135.68	241.59	545.31	2191.10
$s = 2$	77.577	97.057	125.94	170.83	245.61	383.64	682.36	1537.01	6157.74
$s = 3$	151.36	189.87	246.91	335.35	482.45	753.66	1340.16	3017.08	12078.15

Table 12. $n=2$ nodal diameters & $\nu = 0.25$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$s = 0$	5.7822	6.6240	8.1563	10.688	14.949	22.760	39.533	87.398	347.43
$s = 1$	36.968	41.980	50.955	65.933	91.692	139.99	245.40	548.51	2193.04
$s = 2$	90.163	106.09	133.04	176.69	250.58	387.91	686.01	1539.96	6159.40
$s = 3$	166.51	200.06	254.61	341.53	487.58	757.99	1343.80	3019.98	12079.72

Table 13. $n=2$ nodal diameters & $\nu = 0.30$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
s = 0	5.6301	6.4735	8.0070	10.540	14.803	22.623	39.420	87.358	347.65
s = 1	36.941	41.958	50.946	65.944	91.735	140.08	245.58	548.87	2193.91
s = 2	90.184	106.13	133.08	176.76	250.68	388.06	686.24	1540.37	6160.30
s = 3	166.55	200.11	254.68	341.63	487.71	758.17	1344.06	3020.41	12080.65

Table 14. $n=2$ nodal diameters & $\nu = 0.35$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
s = 0	5.4689	6.3153	7.8518	10.387	14.654	22.484	39.307	87.318	347.87
s = 1	36.913	41.937	50.936	65.954	91.777	140.17	245.76	549.22	2194.79
s = 2	90.204	106.16	133.13	176.82	250.78	388.21	686.48	1540.77	6161.21
s = 3	166.59	200.17	254.76	341.73	487.84	758.35	1344.33	3020.84	12081.58

Table 15. $n=3$ nodal diameters & $\nu = 0.25$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
s = 0	12.756	12.924	13.595	15.301	18.925	26.368	43.056	91.089	351.47
s = 1	53.348	55.560	62.209	75.605	100.31	147.88	252.77	555.46	2199.63
s = 2	113.11	122.69	145.75	187.10	259.51	395.80	693.12	1546.48	6165.43
s = 3	194.24	218.34	268.17	352.44	496.79	766.00	1350.93	3026.43	12085.62

Table 16. $n=3$ nodal diameters & $\nu = 0.30$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
s = 0	12.451	12.615	13.276	14.963	18.562	25.980	42.654	90.72	351.32
s = 1	53.203	55.410	62.053	75.447	100.17	147.77	252.73	555.59	2200.28
s = 2	113.04	122.62	145.69	187.06	259.49	395.82	693.23	1546.75	6166.21
s = 3	194.21	218.32	268.16	352.45	496.83	766.09	1351.10	3026.77	12086.46

Table 17. $n=3$ nodal diameters & $\nu = 0.35$

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
s = 0	12.120	12.281	12.932	14.603	18.181	25.579	42.245	90.345	351.17
s = 1	53.057	55.260	61.897	75.291	100.02	147.67	252.70	555.73	2200.94
s = 2	112.97	122.55	145.63	187.01	259.47	395.85	693.34	1547.03	6166.98
s = 3	194.18	218.29	268.15	352.46	496.87	766.18	1351.28	3027.11	12087.30

6. REFERENCES

- [1] A. Leissa, "Vibrations of Plates," NASA SP-160, 1969.
- [2] K. Itao and S. Crandall, "Natural modes and natural frequencies of uniform, circular, free-edge plates," *Journal of Applied Mechanics*, vol. 46, pp. 448-453, 1979.
- [3] T. Irie, G. Yamada and Y. Muramoto, "Free vibrations of a circular annular plate with elastic edge supports," *Journal of Sound and Vibrations*, vol. 71, no. 4, pp. 541-552, 1980.
- [4] P. A. A. Laura, R. H. Gutierrez and R. Rossi, "Vibrations of annular plates with mixed boundary conditions," *Journal of Sound and Vibrations*, vol. 180, no. 5, pp. 741-749, 1995.
- [5] M. V. Predoi, V. Ceausu and E. Deciu, "A comparison of results on circular plates vibrations with various boundary conditions," in *7-th International Congress on Sound and Vibration*, Garmisch-Partenkirchen, Germania, 2000.
- [6] M. V. Predoi and M. Bugaru, "Complementary bending moments influence on the natural frequencies of rotating plates," in *7-th International Congress on Sound and Vibration*, Garmisch-Partenkirchen, Germany, 2000.
- [7] X. Wang and Y. Wang, "Free vibrations analysis of annular plates by the differential quadrature method," *Journal of Sound and Vibrations*, vol. 272, no. 1-2, pp. 273-290, 2004.
- [8] V. Ceausu, N. Enescu, A. Craifaleanu and C. Dragomirescu, "Method for the Study of the Influence of Complex Axial Forces upon the Structural Vibrations," *Romanian Journal of Acoustics and Vibration*, vol. 3, no. 1, pp. 15-18, 2006.
- [9] S. Hosseini-Hashemi and H. Rokni Damavandi Taher, "Exact Solutions for Free Vibration of Annular Mindlin Plates," *International Journal of Mechanical Sciences*, vol. 52, no. 8, pp. 819-829, 2010.
- [10] S. A. Eftekhari and A. A. Jafari, "A Mixed Method for Free and Forced Vibration Analysis of Rectangular and Annular Plates," *Applied Mathematical Modelling*, vol. 36, no. 7, pp. 2814-2831, 2012.
- [11] C. Dragomirescu, A. Craifaleanu and V. Ceausu, "Consideration regarding the nonlinear vibration of a mechanical system," *Romanian Journal of Acoustics and Vibration*, vol. 9, no. 1, pp. 15-22, 2012.
- [12] D. Shi, X. Shi, W. L. Li, Q. Wang and J. Han, "Vibration analysis of annular sector plates under different boundary conditions," *Shock and Vibration*, vol. 2014, pp. 1-11, 2014.
- [13] Y. Zhang and W. L. Li, "Vibration Analysis of Annular Plates with Non-Uniform Thickness," *Journal of Sound and Vibration*, vol. 349, pp. 276-291, 2015.

-
-
- [14] Y. Kumar and R. Lal, "Vibration Analysis of Functionally Graded Annular Plates Using Higher-Order Shear Deformation Theory," *Composite Structures*, vol. 176, pp. 402-414, 2017.
 - [15] F. Farhatnia and M. A. Torkaman-Asadi, "Nonlinear Vibration Analysis of Annular Plates Under Thermal Loading," *Nonlinear Dynamics*, vol. 95, no. 1, pp. 1-18, 2019.
 - [16] D. Zhou, Y. K. Cheung and F. T. K. Au, "Free Vibration Analysis of Annular Plates with Internal Supports," *Thin-Walled Structures*, vol. 146, pp. 106-118, 2020.
 - [17] A. V. Singh and A. K. Sharma, "Vibration Analysis of Annular Plates with Crack Damage Using Finite Element Method," *Engineering Fracture Mechanics*, vol. 245, pp. 107-123, 2021.
 - [18] A. Marinescu, S. Sorohan and T. Cicone, "Accuracy of the analytical and finite element models for circular bending plates," *U.P.B. Sci. Bull., Series D.*, vol. 84, no. 1, pp. 179-192, 2022.
 - [19] J. Wang and Y. Liu, "Machine Learning-Based Prediction of Natural Frequencies of Annular Plates," *Mechanical Systems and Signal Processing*, vol. 168, pp. 108-120, 2022.
 - [20] Z. Chen, Z. Wu, S. Wen and F. Li, "Vibration characteristics of eccentric annular plates," *Thin-Walled Structures*, vol. 191, pp. 1-17, 2023.
 - [21] L. M. Barbu, M. V. Predoi and A. D. Griguța, "Integration of Numerical Simulations in the Study of Vibrations of Circular Plates," in *ISB-INMA-TEH*, Bucharest, 2024.
 - [22] M. V. Predoi, *Vibratii Mecanice - Modele si Aplicatii in Matlab*, Bucharest: Ed. Matrix Rom, 2011, p. 255.
 - [23] "MATLAB, 1994-2025 The MathWorks, Inc.," 2025. [Online]. Available: <https://www.mathworks.com/products/matlab.html>. [Accessed 10 10 2025].
 - [24] COMSOL Multiphysics User Manual, <http://www.comsol.com/>, COMSOL AB. [Online].
 - [25] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, New York: Dover Publications, 1964.